

# IL NUOVO CIMENTO

1954

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PERIODICO ITALIANO DI FISICA

fondato a PISA nel 1885 da C. MATTEUCCI e R. PIRIA

dal 1897 Organo della Società Italiana di Fisica

è pubblicato

sotto gli auspici del Consiglio Nazionale delle Ricerche

a cura del Direttore

GIOVANNI POLVANI

Presidente della Società

e

dei Vicedirettori

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con la collaborazione di un Comitato di redazione

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## Redazione

Bologna, Via Ippolito Niebuhr n. 46  
presso l'Istituto di Fisica dell'Università

## Direzione

Milano, Via Saldini n. 50  
presso l'Istituto di Fisica dell'Università

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# IL NUOVO CIMENTO

A CURA DELLA SOCIETÀ ITALIANA DI FISICA  
SOTTO GLI AUSPICI DEL CONSIGLIO NAZIONALE DELLE RICERCHE

VOLUME XI

*Serie nona*

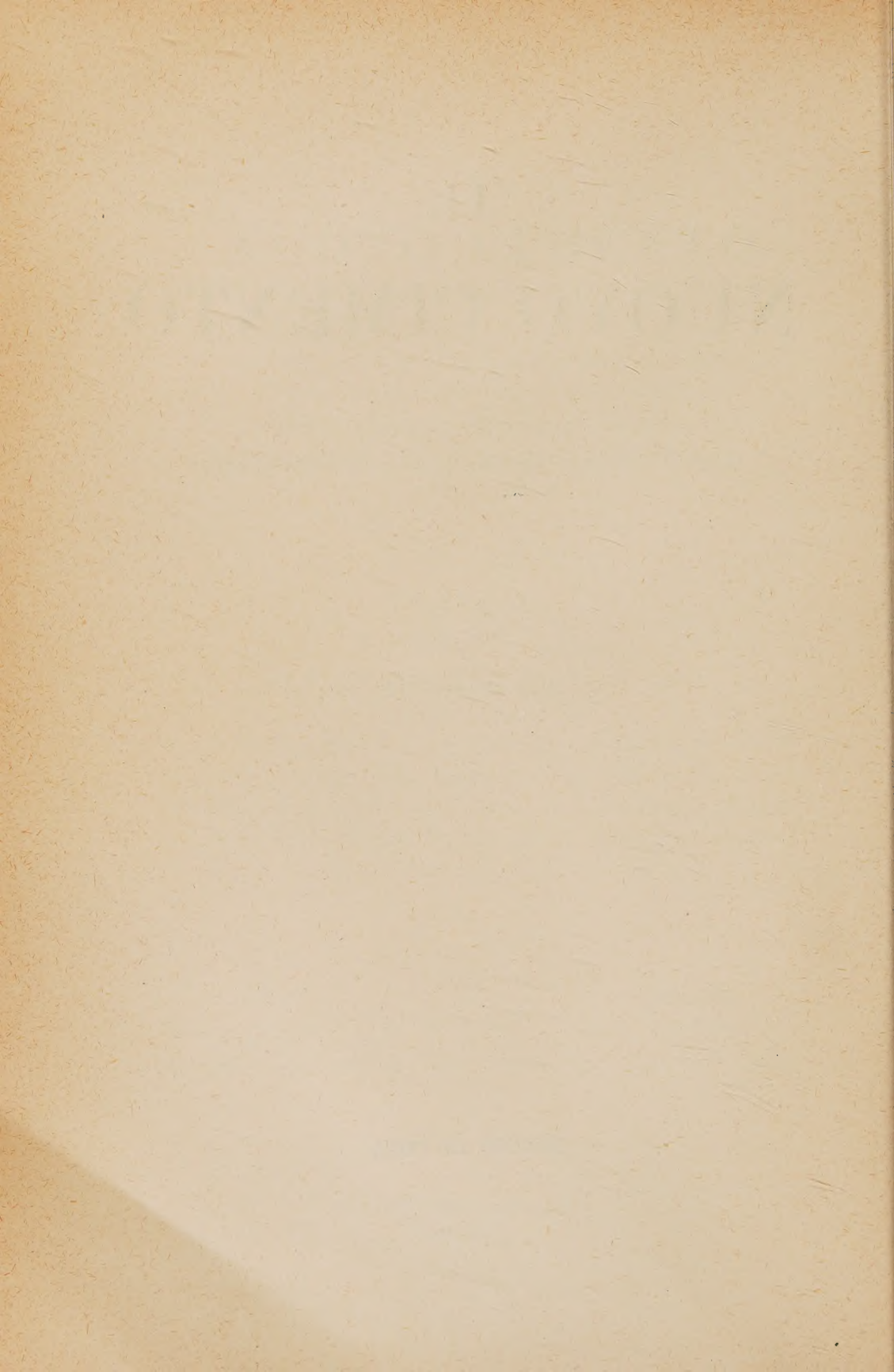
Anno nonagesimonono

1954

PRINTED IN ITALY

NICOLA ZANICHELLI EDITORE  
BOLOGNA





# IL NUOVO CIMENTO

ORGANO DELLA SOCIETÀ ITALIANA DI FISICA

SOTTO GLI AUSPICI DEL CONSIGLIO NAZIONALE DELLE RICERCHE

VOL. XI, N. 1

Serie nona

1° Gennaio 1954

## A Study of the $F \rightarrow Z_1$ Conversion in KCl Crystals with Divalent Impurities.

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(ricevuto il 16 Ottobre 1953)

**Summary.** — The  $F \rightarrow Z_1$  conversion under the action of  $F$  light is studied (Figs. 6 and 7) in KCl crystals containing a small amount of Sr ( $5.5 \cdot 10^{-5}$  molar fraction), which have been colored either additively or by X irradiation. The quantum yield of the conversion during the exposure to  $F$  light has been determined (Figs. 8, 9 and 10). We make the assumption that only the  $\text{Sr}^{++}$  ions which are not associated with a positive ion vacancy are responsible for the formation of  $Z_1$  centers. The general behaviour of the quantum yield during the conversion (Fig. 11) allows one to determine the ratio of the capture cross-sections for electrons of the negative ion vacancies and divalent ions, and the number of negative ion vacancies present in the crystal. The values are respectively  $\sigma_p/\sigma_{Z_1} = \sim 1$  and  $N_v = \sim 1.4 \cdot 10^{17}/\text{cm}^3$ . With the same hypothesis, we can determine the association energy of the complex  $\text{Sr}^{++}$ -positive ion vacancy, which is about of 0,3 eV. From our data we can also determine the value of the oscillator strength of the  $Z_1$  center, which results to be 0.84. The values so obtained for the capture cross-sections, the association energy and the oscillator strength appear entirely reasonable.

### 1. — Introduction.

It is well known that an electron moving in the conduction band of an alkali halide crystal can be trapped at a negative ion vacancy (Fig. 1). The optical transitions between the levels of the center give rise to a characteristic

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absorption band which lies generally in the visible region of the spectrum: the  $F$  band (<sup>1-3</sup>).

PICK (<sup>4</sup>) has discovered several new absorption bands in crystals of KCl and NaCl containing small amounts of alkaline earth ions. These bands, which have been called  $Z$  bands (Zusatzbanden) are certainly due to centers different from those which are responsible for the color centers absorption in pure crystals.

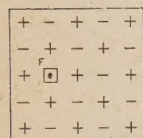


Fig. 1. - The model for the  $F$  center.

Pick's measurements on KCl with divalent impurities show that the behaviour of the various bands is as follows:

1) The additively colored crystals show initially only the normal  $F$  band of the pure crystals.

2) Irradiation of the crystals at room temperature with wavelengths lying in the  $F$  band ( $F$  light), produces the conversion of some of the  $F$  centers into new centers, termed  $Z_1$ , which give rise to an absorption band on the long wavelength side of the  $F$  band.

3) This new band is stable under  $Z_1$  light.

4) The  $Z_1$  band disappears by heating the crystal to temperatures near 110 °C or above: the intensity of the  $F$  band increases and a new band appears (termed  $Z_2$ ) which is thermally stable up to about 250 °C but unstable under his own light.

5) NaCl (<sup>4</sup>) with divalent impurities behaves differently for what concerns point 1) and 4).

SEITZ (<sup>5</sup>) has given a theoretical interpretation of Pick's results and has proposed models for the new centers (Fig. 2).

Here we study the  $F \rightarrow Z_1$  conversion under the action of  $F$  light, at room temperature, in KCl crystals with Sr impurities, colored additively or by X rays at room temperature: our aim was to determine the « quantum yield » of the conversion. These measurements have allowed us to determine also the oscillator strength of the  $Z_1$  center and obtain some information on the capture cross-sections for elec-

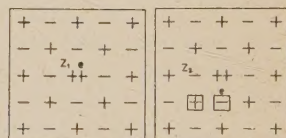


Fig. 2. - The models for the  $Z_1$  and  $Z_2$  centers, after SEITZ.

(<sup>1</sup>) R. W. POHL: *Phys. Zeits.*, **39**, 36 (1938).

(<sup>2</sup>) N. F. MOTT and R. W. GURNEY: *Electronic Processes in Ionic Crystals* (Oxford, 1940), p. 111 and following.

(<sup>3</sup>) F. SEITZ: *Rev. Mod. Phys.*, **18**, 384 (1946).

(<sup>4</sup>) H. PICK: *Ann. d. Phys.*, **35**, 73 (1939); *Zeits. f. Phys.*, **114**, 127 (1939).

(<sup>5</sup>) F. SEITZ: *Phys. Rev.*, **83**, 134 (1951).



trons, of the negative ion vacancy and of the divalent ion and on the association energy of the complex divalent ion — positive ion vacancy.

## 2. — Apparatus.

KCl crystals with a mole fraction of  $\text{SrCl}_2$  equal to about  $5.5 \cdot 10^{-5}$ , were grown by slow cooling from the melt. The Sr content of the crystals was determined by arc spectroscopical analysis. The concentration of other impurities is comparatively negligible.

The crystals were additively colored by baking them in Na vapors at  $600^\circ\text{C}$  for a period of time between 20 minutes and 3 hours. The X rays darkening was achieved by exposing the crystals at room temperature to an ordinary tube operated at 50 kV, 5 mA for a period of time between 10 and 30 minutes, at a distance of about 10 cm from the anticathode.

The measurements of the optical absorption were performed with a photocell spectrophotometer composed by a modulated tungsten lamp, a monochromator, a caesium cell and a narrow band amplifier. The noise voltage was so reduced to less than  $2 \mu\text{V}$  and it was easy to perform the measurements also with very weak intensity of light.

To narrow the color center bands, some of the absorption measurements were performed at liquid air temperature ( $-193^\circ\text{C}$ ) using the cryostat described in Fig. 3: to eliminate the oxygen absorption bands which fall in the region of the  $F$  and  $Z_1$  bands, the light is not made to pass through liquid air.

In order to determine the quantum yield of the  $F \rightarrow Z_1$  conversion it was necessary to know the number of  $F$  light photons impinging on the crystal: the light used was that of the green line of Hg ( $5461 \text{ \AA}$ ). The number of photons was determined by comparing the heatings of a thin blackened plate of manganine produced by the light and by a known current circulating in the lamina. The heating was measured by a thermocouple (Bi-BiSb) attached to the manganine plate. The number of photons was also determined with a bolometer. The two measurements agree within 10%.

## 3. — Results.

*a) The absorption measurements.* — The visible absorption spectrum of a KCl crystal with  $5.5 \cdot 10^{-5}$  mole fraction of  $\text{SrCl}_2$  immediately after additive

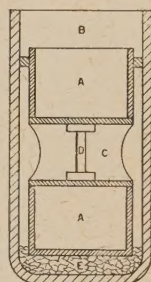


Fig. 3. — The cryostat for making the absorption measurements at the liquid air temperature: A) Brass vessel containing the liquid air. B) Dewar vessel. C) Horizontal tube for passing the light. D) Crystal. E) Desiccant.



coloring is illustrated in Fig. 4: care was taken to avoid exposure of the crystal to light before the measurements, which were performed with a very weak amount of light. Fig. 4 appears to indicate the presence of the *F* band

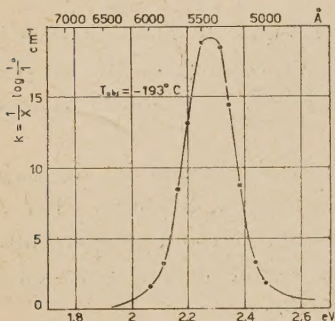


Fig. 4. — Optical absorption spectrum, in the visible, of a KCl crystal containing Sr impurities ( $5.5 \cdot 10^{-5}$  mole fraction) immediately after additive coloring. Observations were performed at liquid air temperature ( $-193^\circ\text{C}$ ).

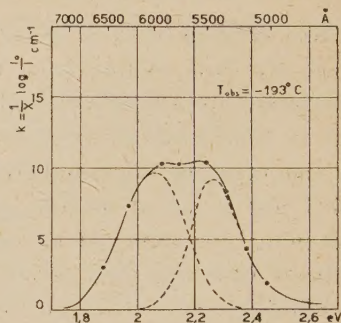


Fig. 5. — Absorption spectrum of the same crystal of fig. 4, observed at liquid air temperature after a long exposure (at room temperature) to *F* light (green line of Hg, 5461  $\text{\AA}$ ).

alone, in agreement with PICK's results (<sup>4</sup>). The visible absorption of crystals colored by X rays at room temperature, immediately after the darkening,

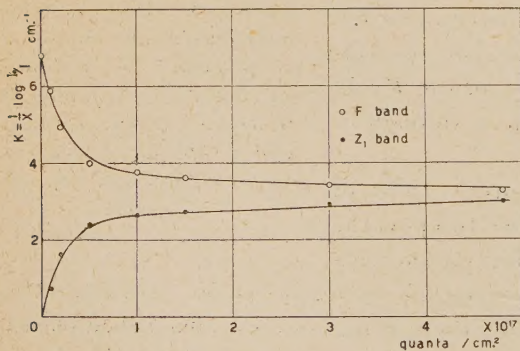


Fig. 6. — Dependence of the maximum absorption coefficients in the *F* and  $Z_1$  bands on the exposure to *F* light. Exposure and observations were made at room temperature.

seems to be the same. Fig. 5 illustrates the absorption spectrum of the crystal of fig. 4 after a long exposure to the green line of Hg: one notices the presence of the  $Z_1$  band at the low energy side of the *F* band. No other band appears in relevant quantity in the visible and near infrared spectrum: in particular the *R* and *M* bands formed after prolonged illumination are quite small.

The detailed variations of the *F* and  $Z_1$  bands during the exposure to *F* light at room

temperature are illustrated in Figs. 6 and 7, respectively for an additively and an X rays colored crystal. Fig. 6 shows that the absorption coefficients at the maximum of the two bands tend to a limiting constant value, while Fig. 7



shows that the absorption coefficients tend to zero, as one would expect, in an X rays colored crystal.

In an additively colored crystal, the conversion of the  $F$  band in  $Z_1$  band is generally not total: one can observe the total conversion only if the initial  $F$  band is small enough.

Table I gives the maxima and half widths of the  $F$  and  $Z_1$  bands at room temperature and liquid air temperature. The proportionality between the half width and the square root of the temperature is obeyed <sup>(6)</sup>.

b) *The « oscillator strength » of the  $Z_1$  center.* — The absorption measurements of section 3a) indicate that, in a crystal additively colored, electrons freed from  $F$  centers by  $F$  light form practically either  $Z_1$  or  $F$  centers.

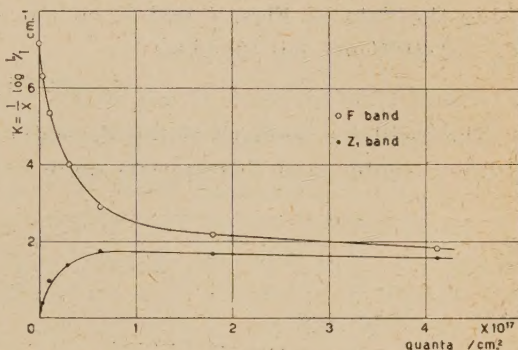


Fig. 7. — The same as fig. 6 for a crystal colored by X rays.

TABLE I.

	$T_{\text{obs.}}$	$h\nu_{\text{max}}$ (in eV)	$W$ (in eV)
$F$ band	80 °K	2.28	0.20
	293 °K	2.19	0.38
$Z_1$ band	80 °K	2.06	0.25
	293 °K	2.00	0.46

The number  $\Delta N_F$  of  $F$  centers destroyed per  $\text{cm}^3$  is then equal to the number of  $Z_1$  centers formed per  $\text{cm}^3$   $N_{Z_1}$ .  $\Delta N_F$  can be calculated from the variation of the  $F$  band, using Smakula's formula <sup>(7)</sup>; for KCl we have:

$$(1) \quad \Delta N_F = 1.06 \cdot 10^{16} \cdot \frac{W}{f} \Delta k_{\text{max}} \quad \text{per cm}^3,$$

where  $W$  is the width at half maximum of the  $F$  band in eV,  $\Delta k_{\text{max}}$  is the variation of the absorption coefficient at the maximum and  $f$  is the oscillator

<sup>(6)</sup> Ref. <sup>(2)</sup>, p. 117.

<sup>(7)</sup> A. SMAKULA: *Zeits. f. Phys.*, **59**, 603 (1930).

strength of the  $F$  center which is known to be 0.81 <sup>(8)</sup>. In the same way:

$$(2) \quad N_{Z_1} = 1.06 \cdot 10^{16} \frac{W}{f_{Z_1}} k_{Z_1 \max}.$$

Using the data of Figs. 4 and 5, eq. (1) yields  $\Delta N_F = 3 \cdot 10^{16}/\text{cm}^3$ . Putting  $N_{Z_1} = 3 \cdot 10^{16}/\text{cm}^3$ , eq. (2) gives

$$f_{Z_1} = 0.84.$$

The oscillator strength of the  $Z_1$  center in KCl is then comparable to that of the  $F$  center, and this appears quite reasonable owing to the similarity of the two types of centers that SEITZ has pointed out <sup>(5)</sup>.

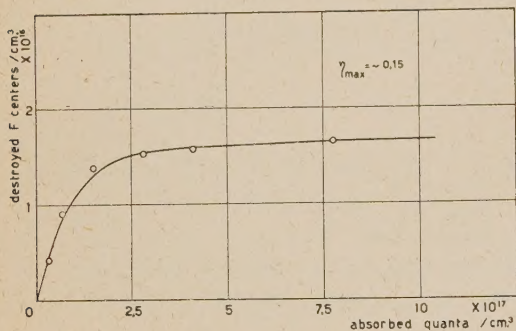


Fig. 8. — Detailed variation of the numbers of  $F$  and  $Z_1$  centers as a function of the number of the  $F$  photons absorbed, for a crystal colored additively. The slope of the curve is the quantum yield for the  $F \rightarrow Z_1$  conversion.

*c) The quantum yield of the  $F \rightarrow Z_1$  conversion.* — The results plotted in Figs. 6 and 7 allow us to determine the quantum yield of the  $F \rightarrow Z_1$  conversion in additively and X rays colored crystals using the values of the oscillator strengths for  $F$  and  $Z_1$  centers. This is shown in Figs. 8, 9 and 10 which give the variation of the number of  $F$  centers and the number of  $Z_1$  centers per  $\text{cm}^3$ ,  $\Delta N_F$  and  $N_{Z_1}$ ,

as a function of the number of  $F$  photons absorbed  $N_{hv}$ : for X rays colored

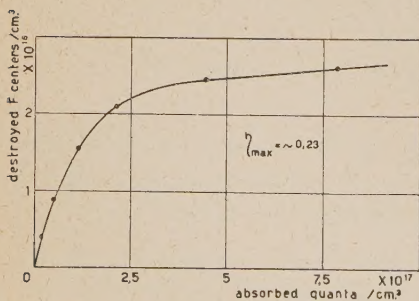


Fig. 9. — Variation of the number of  $F$  centers as a function of the number of the  $F$  photons absorbed in a crystal colored by X rays. The slope is the quantum yield for the destruction of the  $F$  centers.

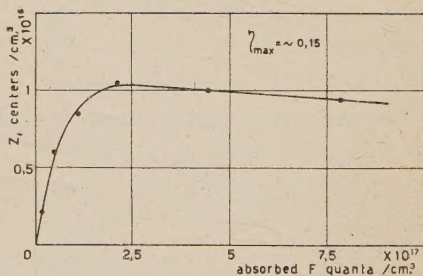


Fig. 10. — Variation of the number of  $Z_1$  centers as a function of the number of  $F$  photons absorbed for the same crystal of fig. 9. The slope is the quantum yield for the formation of  $Z_1$  centers.

<sup>(8)</sup> F. G. KLEINSCHROD: *Ann. d. Phys.*, **27**, 97 (1936).



crystals  $\Delta N_F$  is obviously different from  $N_{Z_1}$ . The slope of the curves gives the quantum yield.

#### 4. - Discussion of the results.

The two striking facts in the behaviour of the  $Z_1$  band are its absence in the freshly colored crystals and its saturation in the  $F \rightarrow Z_1$  conversion.

a) *The saturation of the  $Z_1$  band in the  $F \rightarrow Z_1$  conversion.* - The lack of a backward reaction in the  $F \rightarrow Z_1$  conversion owing to the stability of the  $Z_1$  centers under  $Z_1$  light indicates that the limiting value of the  $Z_1$  band is due to saturation<sup>(9)</sup>. Furthermore at saturation only a number of  $\text{Sr}^{++}$  of the order of  $10^{16}/\text{cm}^3$  have captured an electron, while about  $10^{18}$   $\text{Sr}^{++}$  per  $\text{cm}^3$  were present in our crystals. The only reasonable explanation we could find is that only  $\text{Sr}^{++}$  which are not associated with positive ion vacancy have an appreciable capture cross-section for conduction electrons and that the largest part of  $\text{Sr}$  ions are associated with a positive ion vacancy. The mean life of this complexes at room temperature is of the order of  $10^4$  s if one considers the values of the association energy and of the jump frequency for free positive ion vacancy.

We have exploited the consequence of this hypothesis determining with it the ratio of the capture cross-sections for electrons of a negative ion vacancy  $\sigma_F$  and of a free  $\text{Sr}^{++}$   $\sigma_{Z_1}$ , as well as the association energy  $W$  of  $\text{Sr}^{++}$  with a positive ion vacancy, to see if we obtain reasonable values. With our calculations we determine values of  $\sigma_F/\sigma_{Z_1}$  and of  $W$  which are respectively an upper and a lower bound. The main assumptions involved are that  $++\square$  complexes have a zero capture cross-section at room temperature<sup>(10)</sup>, that the dissociation of the complexes is purely thermal and that only  $F$  and  $Z_1$  centers are formed during the  $F \rightarrow Z_1$  conversion.

b) *The capture cross-section for electrons of the free  $\text{Sr}^{++}$ .* - If we assume that each photon absorbed in the  $F$  band frees an electron from an  $F$  center<sup>(11-13)</sup>, the quantum yield of the  $F \rightarrow Z_1$  conversion in additively colored crystals can be written

$$(3) \quad \eta = - \frac{d\Delta N_F}{dN_{h\nu}} = \frac{dN_{Z_1}}{dN_{h\nu}} = \frac{dN_{h\nu} \tau (N^{++} - N_{Z_1}) \sigma_{Z_1} u}{dN_{h\nu}},$$

<sup>(9)</sup> This is confirmed by the total conversion in  $Z_1$  band of a small  $F$  band.

<sup>(10)</sup> H. U. HARTEN: *Nachr. Akad. Wiss. Göttingen*, **15** (1950).

<sup>(11)</sup> Ref. (2), p. 129 and 136.

<sup>(12)</sup> H. PICK: *Ann. d. Phys.*, **31**, 365 (1938).

<sup>(13)</sup> R. W. POHL: *Proc. Phys. Soc.*, **49**, 13 (1937).

where  $u$  is the thermal velocity of the conduction electrons,  $\tau$  their mean life,  $N^{++}$  the number of free  $\text{Sr}^{++}/\text{cm}^3$ ; the other symbols have already been defined. Introducing in eq. (3) the mean life

$$(4) \quad \tau = \frac{1}{u[(N_v - N_F)\sigma_F + (N^{++} - N_{Z_1})\sigma_{Z_1}]},$$

where  $N_v$  is the number of negative ion vacancies per  $\text{cm}^3$ , and remembering that  $N_F = N_F^0 - N_{Z_1}$ , we obtain

$$(5) \quad \eta = \frac{(N^{++} - N_{Z_1})\sigma_{Z_1}}{(N_v - N_F^0 + N_{Z_1})\sigma_F + (N^{++} - N_{Z_1})\sigma_{Z_1}}.$$

Fig. 11 shows the dependence of  $\eta$  on  $N_{Z_1}$  in a crystal additively colored. Clearly this dependence is nearly linear. From (5) (which is the equation of an hyperbola) we can see that this is possible only if  $\sigma_F \cong \sigma_{Z_1}$ . Putting  $N^{++} = N_{Z_1\text{max}} = 1.8 \cdot 10^{16}/\text{cm}^3$ , in agreement with our hypothesis, and  $\sigma_F = \sigma_{Z_1}$ , according to the latter result, we find the number of negative ion vacancies  $N_v = 1.4 \cdot 10^{17}/\text{cm}^3$ . These values appear quite reasonable.

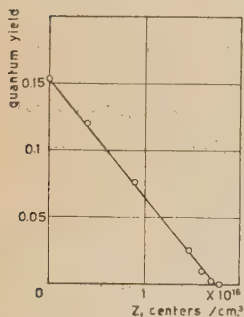
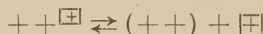


Fig. 11. - Dependence of the quantum yield on the actual number of  $Z_1$  centers present during the conversion, in a crystal additively colored.

c) The association energy of the complex  $++\boxed{+}$ . - The equilibrium constant  $K$  of the reaction



is given by

$$(6) \quad K = \frac{x_c}{(y - x_c)(x_p - x_c)}$$

( $y$  = total molar fraction of Sr;  $x_c$  = molar fraction of associated complexes;  $x_p$  = molar fraction of the positive ion vacancies<sup>(14)</sup>, and is related to the heat of reaction  $W$  (association energy of the complex) by the equation<sup>(3-15)</sup>)

$$(7) \quad K = 12 \exp [W/kT].$$

Equation (7) enables us to compute the energy  $W$  if one can compute  $K$  from eq. (6). If we put  $x_c = y - x_{Z_1\text{max}}$  in agreement with our hypothesis on the formation of the  $Z_1$  centers, and use the value of  $N_v$  found in section 4b) and

(14) For the neutrality of the crystal, one requires that  $x_p = x_v + y$  where  $x_v$  is the molar concentration of the negative ion vacancies.

(15) A. BEAN: *Thesis Un. Illinois* (1952, to be published).



$y = 5.5 \cdot 10^{-5}$   $T = \sim 300$  °K <sup>(16)</sup>, we obtain:

$$W = \sim 0.3 \text{ eV}.$$

This value agrees well with the theoretical value <sup>(17)</sup> and is of the same order of magnitude of the association energies of similar complexes <sup>(18)</sup>.

d) *The behaviour of the  $Z_1$  band in X rays colored crystals.* - The initial absence of  $Z_1$  band in additively colored crystals is easily understood if one considers that the  $Z_1$  center is thermally instable in KCl above 110 °C, while the  $F$  center is not.

A total absence of  $Z_1$  band in freshly X rays colored crystals instead seems to be inexplicable. In our experimental observations the height of the  $Z_1$  band seems to be zero or too small to be observed experimentally. The initial quantum yield for the formation of the  $Z_1$  centers in the  $F \rightarrow Z_1$  conversion for X rayd crystals is 0.15 but is not clear how significant this figure is in the primary X rays darkening, since one does not know the relative numbers of traps present in the two cases. The presence of an initial  $Z_1$  band too small to be observed experimentally may be rationalized if one considers that the capture cross sections for electrons of a negative ion vacancy and of a free  $\text{Sr}^{++}$  are of the same order of magnitude and that the X rays darkening introduces vacancies <sup>(19)</sup>. This explanation does not disagree with the equality between the value 0.15 we find for the initial quantum yield for the formation of  $Z_1$  centers in X rays colored crystals and the value of the initial yield in chemically colored crystals, since the vacancies introduced by X rays probably cluster.

An X rays colored crystal is totally bleached after long exposure to light (Fig. 7). The bleaching of the  $Z_1$  band could be caused by a small chance that the electron jumps in the conduction band when a photon is absorbed. Another process could be the recombination of  $V$  and excited  $Z_1$  centers: this process could occur also between two centers which are at a relatively great distance.

The authors wish to express their thanks to Profs. L. GIULOTTO and F. FUMI for very helpful discussions, advices and continued interest. They are

<sup>(16)</sup> The choice of  $T = \sim 300$  °K is perhaps not entirely correct, since the equilibrium corresponding to higher temperatures may be frozen in the crystal.

<sup>(17)</sup> F. BASSANI e F. FUMI: *Nuovo Cimento* (to be published).

<sup>(18)</sup> H. ETZEL and R. J. MAURER: *Journ. Chem. Phys.*, **18**, 1003 (1950); O. STASIW and J. TELTOW: *Ann. d. Phys.* **1** 261 (1947).

<sup>(19)</sup> I. ESTERMANN, W. J. LEIVO and O. STERN, *Phys. Rev.*, **74**, 627 (1949); K. SAKAGUCHI and T. SUITA: *Tech. Rep. Osaka Univ.* **2**, 177 (1952).

also indebted to Prof. F. SEITZ for useful comments. This work has been performed with the financial support of the Consiglio Nazionale delle Ricerche.

## RIASSUNTO

Viene studiata la conversione  $F \rightarrow Z_1$  (Fig. 6 e 7) sotto l'azione di luce  $F$ , in cristalli di KCl contenenti una piccola concentrazione di Sr ( $5.5 \cdot 10^{-5}$  frazioni molari), colorati sia additivamente sia mediante esposizione a raggi X. Viene determinato il rendimento quantico della conversione  $F \rightarrow Z_1$  (Fig. 8, 9 e 10). Viene fatta l'ipotesi che solo gli ioni  $\text{Sr}^{+1}$  che non sono associati ad una vacanza di ione negativo siano responsabili della formazione dei centri  $Z_1$ . Dall'andamento della resa quantica durante la conversione è stato possibile determinare il rapporto tra le sezioni di cattura per elettroni della vacanza di ione negativo e dello ione bivalente libero oltre che il numero di vacanze di ione negativo presenti nel cristallo. I valori trovati sono rispettivamente  $\sigma_F/\sigma_{Z_1} = \sim 1$  e  $N_v = \sim 1.4 \cdot 10^{17}/\text{cm}^3$ . Sempre in base alla stessa ipotesi si è potuto determinare l'energia di associazione del complesso vacanza di ione negativo-ione bivalente libero, che è risultata uguale a circa 0,3 eV. Dai nostri dati è stato anche possibile determinare il valore della « oscillator strength » del centro  $Z_1$ : esso è risultato di 0,84. I valori ottenuti per le sezioni di cattura, l'energia di associazione e l'« oscillator strength » sembrano completamente ragionevoli.



## Solution of a Non-Isotropic Random Flight Problem in the Case of a Non-Isotropic Point Source.

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(ricevuto il 20 Ottobre 1953)

**Summary.** — In connection with the calculation of the perturbation caused by a thin circular detector in a thermal neutron flux, we have solved the following problem. Given is a point source whose strength is described by an isotropic distribution plus a non-isotropic cosine term. The single scattering law also contains one non-isotropic term. We have to calculate the neutron density and the distribution of the velocity directions in every point of space. Keeping the path probabilities arbitrary for the reason of generality, we have established exact recurrence relations between the defined probability distributions. Expanding these functions in series of spherical harmonics and making use of addition theorems involving Bessel functions and Legendre polynomials, it has been possible to transform the integral relations to simple linear equations. The final results are obtained in the form of integrals which can be calculated by numerical evaluation using computing machines. By introducing simplified functions under the integral signs, it is also possible to obtain good approximations for the neutron density and current. A simple example of this method is discussed in detail and the results are compared with those given by diffusion theory.

### Introduction.

Studying the depression of the neutron density in the vicinity of a thin circular detector placed in a spherically symmetric thermal neutron distribution, one finds that the points of the detector space act like negative non-

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isotropic point sources. Their strength can be sufficiently well described by the isotropic term and the first non-isotropic term in its Legendre expansion. In order to give an accurate description of the density depression, it is clear that one needs the multiple scattering theory of particles emitted by such non-isotropic point sources. The particles in question are thermal neutrons which are almost isotropically scattered in the medium. We could have calculated the formulae on the basis of purely isotropic scattering by the introduction of the transport mean free path. However, for various reasons, we have preferred to derive our final formulae from a more rigorous theory in which the angular distribution of the scattered neutrons is corrected up to the first order of non-isotropy. In this way we are led to the quite difficult problem of finding the particle density around a non-isotropic point source, when the single scattering law contains itself one non-isotropic term. In the next paragraphs, it will be shown how this interesting random flight problem can be treated using a procedure based on recurrence relations which we have already used before to solve some problems of this kind <sup>(1)</sup>.

### Statement of the Problem.

Suppose there is a non-isotropic point source present in the origin  $O$  of a coordinate-system  $Oxyz$  (Fig. 1). The number of particles emitted per unit time between  $\alpha$  and  $\alpha + d\alpha$  with respect to the  $z$ -axis, is given by

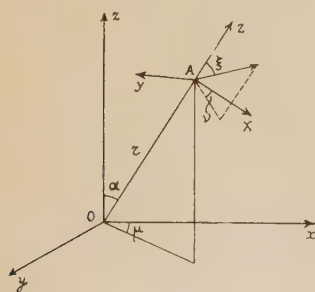


Fig. 1.

$$(1) \quad S_0(1 + A \cos \alpha) \frac{\sin \alpha d\alpha}{2}$$

where  $S_0$  is the total source strength.

The emitted particles diffuse through the medium, undergoing a number of scattering collisions. Let  $f(l)dl$  represent the elementary probability that after a scattering, a particle would cover a distance between  $l$  and  $l + dl$  before it is again scattered. As we are in fact solving a well-defined mathematical random flight problem, we wish to keep the function  $f(l)$  arbitrary. Of course, one condition has to be satisfied: if no capture is possible, the distribution  $f(l)dl$  must be normalized to unity

$$(2) \quad \int_0^{\infty} f(l) dl = 1,$$

<sup>(1)</sup> C. C. GROSJEAN: *Physica*, **19**, 29 (1953); *Verh. Kon. Vl. Acad. Wetensch. (Belgium)*, **13**, n. 36 (1951).



whereas in the case of capture, we get

$$(2') \quad \int_0^{\infty} f(l) dl < 1.$$

For thermal neutrons, it is known that

$$(3) \quad f(l) dl = \exp \left[ -\frac{l}{\lambda} \right] \frac{dl}{\lambda_s}$$

where  $\lambda_s$ ,  $\lambda_c$  and  $\lambda$  are respectively the scattering, capture and total mean free paths, and where  $\lambda$  is defined by

$$(4) \quad \frac{1}{\lambda} = \frac{1}{\lambda_s} + \frac{1}{\lambda_c}.$$

All these mean free paths are supposed to be the properly chosen averages over the thermal energy region.

Let

$$(5) \quad p(\gamma) \frac{\sin \gamma d\gamma}{2} = (1 + 3b \cos \gamma) \frac{\sin \gamma d\gamma}{2}$$

represent the elementary probability that a particle would be deviated over an angle between  $\gamma$  and  $\gamma + d\gamma$  during a scattering collision. It is clear that the meaning of  $b$  is

$$(6) \quad b \equiv \langle \cos \gamma \rangle_{av}.$$

Due to the fact that we shall have to deal with angular distributions of velocity vectors in each point  $A$  of space, we need a «local» frame of reference with  $A$  as origin, to associate two angular coordinates to each direction. If  $A$  is described by the spherical polar coordinates  $(r, \alpha, \mu)$ , the local system can be defined as follows:

- a local  $AZ$ -axis along  $OA$  in the direction corresponding to an increasing  $r$ -coordinate;
- a local  $AX$ -axis perpendicular to  $OA$  and in the plane  $zOA$ , in the direction corresponding to an increasing  $\alpha$ -coordinate;
- a local  $AY$ -axis perpendicular to  $zOA$  and in a direction corresponding to an increasing  $\mu$ -coordinate.

Each direction through  $A$  can now be described by the usual polar coordinates  $\xi$  and  $\nu$  (Fig. 1). In the points of the  $Oz$ -axis, the local frames of

reference are not completely determined by the previous definitions, but due to the axial symmetry in the problem we do not strictly need the local coordinate systems in these points.

Now, we define:

$$\varphi_n(r, \alpha; \xi, \nu) dr \frac{\sin \alpha d\alpha}{2} \cdot \frac{\sin \xi d\xi d\nu}{4\pi} :$$

as the elementary probability that a particle emitted by the source, would undergo the  $n$ -th scattering in the ring-element  $(r, r+dr)(\alpha, \alpha+d\alpha)$  and that it would be moving before that collision in the elementary solid angle  $(\xi, \xi+d\xi)(\nu, \nu+d\nu)$  with respect to the local frame of reference at the scattering point.

$$\psi_n(r, \alpha; \xi, \nu) dr \frac{\sin \alpha d\alpha}{2} \cdot \frac{\sin \xi d\xi d\nu}{4\pi} :$$

as the elementary probability that a particle emitted by the source, would undergo the  $n$ -th scattering in the ring-element  $(r, r+dr)(\alpha, \alpha+d\alpha)$  and that it would continue its trajectory in the elementary solid angle  $(\xi, \xi+d\xi)(\nu, \nu+d\nu)$  with respect to the local frame of reference at the scattering point.

The problem consists in calculating the exact mathematical forms of  $\varphi_n$  and  $\psi_n$ , so that by summation with respect to  $n$ , we can find the total distributions

$$(7) \quad \left\{ \begin{array}{l} \Phi(r, \alpha; \xi, \nu) = \sum_{n=1}^{\infty} \varphi_n(r, \alpha; \xi, \nu) , \\ \Psi(r, \alpha; \xi, \nu) = \sum_{n=1}^{\infty} \psi_n(r, \alpha; \xi, \nu) . \end{array} \right.$$

In the physical case, the function  $\Phi$  is directly related to the particle density  $\varrho(r, \alpha; \xi, \nu)$ .

### Theoretical Development.

There is no real difficulty in finding the functions related to the first scattering collision:

$$(8) \quad \left\{ \begin{array}{l} \varphi_1(r, \alpha; \xi, \nu) = \frac{2\delta(\xi)f(r)(1+A \cos \alpha)}{\sin \xi} , \\ \psi_1(r, \alpha; \xi, \nu) = f(r)(1+A \cos \alpha)(1+3b \cos \xi) , \end{array} \right.$$

where  $\delta(\xi)$  is the Dirac  $\delta$ -function.

Our method consists in establishing recurrence relations between the defined probabilities  $\varphi_n$  and  $\psi_n$ , and solving them exactly by the use of their



appropriate integral representations. The first recurrence relation involves the angular distributions of the particles before and after their  $n$ -th scattering in an arbitrary point of space. The probability that the incident direction lies in  $d\Omega' = \sin \xi' d\xi' d\nu'$  is multiplied by the probability for a scattering into  $d\Omega = \sin \xi d\xi d\nu$  and the product is integrated over all the directions  $(\xi', \nu')$ :

$$(9) \quad \psi_n(r, \alpha; \xi, \nu) = \iint \varphi_n(r, \alpha; \xi', \nu') (1 + 3b \cos \gamma) \frac{d\Omega'}{4\pi} \quad (n = 1, 2, \dots)$$

where  $\cos \gamma = \cos \xi \cos \xi' + \sin \xi \sin \xi' \cos (\nu - \nu')$ .

This can be integrated using the expansion of  $\varphi_n$  in a series of spherical harmonics. Due to the symmetry of the problem with respect to each  $zOA$ -plane, the distributions  $\varphi_n$  are even functions of  $\nu$ , so that their Fourier expansion with respect to  $\nu$  will be a cosine series:

$$(10) \quad \varphi_n(r, \alpha; \xi, \nu) = \sum_{k=0}^{\infty} \sum_{m=0}^k \varphi_n^{(k,m)}(r, \alpha) P_k^m(\cos \xi) \cos m\nu \quad (n = 1, 2, \dots).$$

Introducing (10) in (9) and taking into account the orthogonality properties of the spherical harmonics, it is easy to find that

$$(11) \quad \begin{aligned} \psi_n(r, \alpha; \xi, \nu) = & \varphi_n^{(0,0)}(r, \alpha) + b\varphi_n^{(1,0)}(r, \alpha) \cos \xi + \\ & + b\varphi_n^{(1,1)}(r, \alpha) \sin \xi \cos \nu \quad (n = 1, 2, 3, \dots). \end{aligned}$$

For  $n = 1$ , the expansion of  $\varphi_1$  leads us to

$$(12) \quad \begin{cases} \varphi_1^{(k,0)}(r, \alpha) = (2k + 1)f(r)(1 + A \cos \alpha) & (k = 0, 1, 2, \dots), \\ \varphi_1^{(k,m)}(r, \alpha) = 0 & (m = 1, 2, \dots), \end{cases}$$

and looking at the expression for  $\psi_1$ , we see that (11) is satisfied. The second recurrence relation can be established as follows:

— let us consider the locus of point  $B$ , for which the direction of the vector  $BA$  is described by the polar coordinates  $(\xi, \nu)$  in the local frame of reference of  $A$  (Fig. 2);

— the probability that a particle would undergo the  $(n-1)$ -th scattering in  $dV'$  around  $B$  and

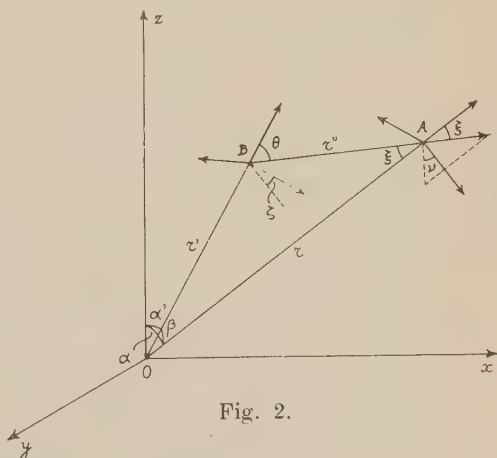


Fig. 2.

that it would continue its path in the elementary solid angle  $(\theta, \theta + d\theta)(\zeta, \zeta + d\zeta)$  is

$$(13) \quad \frac{\psi_{n-1}(r', \alpha'; \theta, \zeta)}{4\pi r'^2} dV' \frac{\sin \theta d\theta d\zeta}{4\pi};$$

— multiplying this by  $f(r'') dr''$ , we get the probability that the  $(n-1)$ -th scattering happens in  $dV'$  and the  $n$ -th scattering in  $dV$  around  $A$ :

$$(14) \quad \frac{\psi_{n-1}(r', \alpha'; \theta, \zeta)}{4\pi r'^2} dV' \frac{\sin \theta d\theta d\zeta}{4\pi} f(r'') dr'' = \frac{\psi_{n-1}(r', \alpha'; \theta, \zeta)}{4\pi r'^2} \cdot \frac{f(r'')}{4\pi r''^2} dV dV',$$

where  $dV = r''^2 \sin \theta dr'' d\theta d\zeta$  is the volume-element around  $A$ , described in the local frame of  $B$ . Going over to the local frame of  $A$ , we see that  $dV' = r''^2 \sin \xi dr'' d\xi dv$  and therefore, we get for (14):

$$(15) \quad \frac{\psi_{n-1}(r', \alpha'; \theta, \zeta)}{4\pi r'^2} f(r'') dr'' dV \frac{\sin \xi d\xi dv}{4\pi};$$

— integrating this over all possible values of  $r''$ , namely from zero to infinity, we must find the probability that the  $n$ -th scattering occurs in  $dV$  around  $A$ , with an incoming direction in  $(\xi, \xi + d\xi)(v, v + dv)$ . Hence

$$(16) \quad \boxed{\varphi_n(r, \alpha; \xi, v) = r^2 \int_0^\infty \frac{\psi_{n-1}(r', \alpha'; \theta, \zeta)}{r'^2} f(r'') dr''} \quad (n = 2, 3, \dots),$$

in which  $[\psi_{n-1}(r', \alpha'; \theta, \zeta)]/r'^2$  is an implicit function of the variables  $r, \alpha, \xi, v$  and  $r''$ .

Making use of (11), we obtain a new recurrence relation, involving the components of the functions  $\varphi_n$  only:

$$(17) \quad \varphi_n(r, \alpha; \xi, v) = r^2 \int_0^\infty \frac{\varphi_{n-1}^{(0,0)}(r', \alpha')}{r'^2} f(r'') dr'' + br^2 \int_0^\infty \frac{\varphi_{n-1}^{(1,0)}(r', \alpha')}{r'^2} \cos \theta f(r'') dr'' + \\ + br^2 \int_0^\infty \frac{\varphi_{n-1}^{(1,1)}(r', \alpha')}{r'^2} \sin \theta \cos \zeta f(r'') dr'',$$

which we have to solve starting with the known components:

$$(18) \quad \varphi_1^{(0,0)}(r, \alpha) = f(r)(1 + A \cos \alpha); \quad \varphi_1^{(1,0)}(r, \alpha) = 3f(r)(1 + A \cos \alpha); \quad \varphi_1^{(1,1)}(r, \alpha) = 0.$$



**Solution of  $\varphi_2(r, \alpha; \xi, \nu)$ .**

Let us apply (17) for  $n = 2$ :

$$(19) \quad \varphi_2(r, \alpha; \xi, \nu) = r^2 \int_0^\infty \frac{f(r')f(r'')}{r'^2} (1 + A \cos \alpha') dr'' + \\ + 3br^2 \int_0^\infty \frac{f(r')f(r'')}{r'^2} (1 + A \cos \alpha') \cos \theta dr''.$$

Using the relations between the elements of the trihedral  $OzAB$  (Fig. 2), we can write

$$(20) \quad \cos \alpha' = \cos \alpha \cos \beta + \sin \alpha \sin \beta \cos \nu$$

and we get for (19)

$$(21) \quad \varphi_2(r, \alpha; \xi, \nu) = r^2 \int_0^\infty \frac{f(r')f(r'')}{r'^2} (1 + 3b \cos \theta) dr'' + \\ + Ar^2 \cos \alpha \int_0^\infty \frac{f(r')f(r'')}{r'^2} (1 + 3b \cos \theta) \cos \beta dr'' + \\ + Ar^2 \sin \alpha \cos \nu \int_0^\infty \frac{f(r')f(r'')}{r'^2} (1 + 3b \cos \theta) \sin \beta dr'',$$

in which the three integrals contain only functions of the elements of the triangle  $OAB$ .

The remaining problem consists in finding the Legendre series expansions of these integrals. This can be done as follows. The form of the integrals in (21) suggests the use of certain addition theorems involving Bessel functions and Legendre polynomials. It is well-known that by means of such addition theorems, it becomes possible to obtain series expansions of integrals, in which the functions are depending on various elements of a triangle. Let us for instance consider the simplest addition theorem, which we shall apply:

$$(22) \quad \frac{\sin r'u}{r'u} = \sum_{k=0}^{\infty} (2k+1) P_k(\cos \xi) \frac{j_{k+\frac{1}{2}}(ru)}{ru} \cdot \frac{j_{k+\frac{1}{2}}(r''u)}{r''u},$$

in which  $j_{k+\frac{1}{2}}(x)$  is defined by

$$(23) \quad j_{k+\frac{1}{2}}(x) \equiv \sqrt{\frac{\pi x}{2}} J_{k+\frac{1}{2}}(x).$$

Let us suppose that we can write  $(1/r)f(r)$  as a Fourier integral:

$$(24) \quad \frac{f(r)}{r} = \int_0^{\infty} \mathcal{F}(u) \sin ru \, du$$

and let us introduce this into the integral

$$(25) \quad \int_0^{\infty} \frac{f(r')f(r'')}{r'^2} \, dr'',$$

then we can apply (22) in the following way

$$\begin{aligned} (26) \quad \int_0^{\infty} \frac{f(r')f(r'')}{r'^2} \, dr'' &= \int_0^{\infty} \frac{f(r'')}{r'} \, dr'' \int_0^{\infty} \mathcal{F}(u) \sin r'u \, du = \\ &= \int_0^{\infty} f(r'') \, dr'' \int_0^{\infty} u \mathcal{F}(u) \left( \frac{\sin r'u}{r'u} \right) \, du = \\ &= \frac{1}{r} \sum_{k=0}^{\infty} (2k+1) P_k(\cos \xi) \int_0^{\infty} \mathcal{F}(u) j_{k+\frac{1}{2}}(ru) \, du \int_0^{\infty} f(r'') \frac{j_{k+\frac{1}{2}}(r''u)}{r''u} \, dr''. \end{aligned}$$

In our previous research in the field of multiple scattering, we have introduced the symbol:

$$(27) \quad F^{(k,m)}(u) = \int_0^{\infty} f(y) \left[ i^m P_m \left( \frac{d}{i \, dz} \right) \frac{j_{k+\frac{1}{2}}(z)}{z} \right]_{z=yu} \, dy,$$

where  $P_m(d/i \, dz)$  is a differential operator obtained by replacing each power  $x^s$  by  $(1/i^s)(d^s/dz^s)$  in the Legendre polynomial  $P_m(x)$ . Using the inversion formula of (24) to find  $\mathcal{F}(u)$  at least formally, we get

$$(28) \quad \mathcal{F}(u) = \frac{2}{\pi} \int_0^{\infty} \frac{f(r)}{r} \sin ru \, dr = \frac{2u}{\pi} F^{(0,0)}(u)$$



and putting this into (26), we finally obtain:

$$(29) \quad \int_0^{\infty} \frac{f(r')f(r'')}{r'^2} dr'' = \frac{2}{\pi r} \sum_{k=0}^{\infty} (2k+1) P_k(\cos \xi) \int_0^{\infty} u F^{(0,0)}(u) F^{(k,0)}(u) j_{k+\frac{1}{2}}(ru) du,$$

which is precisely the expansion which we wished to obtain.

An apparent difficulty in this mathematical scheme is related to the convergence of the Fourier integral (24). It may happen that this integral is oscillating divergent although  $F^{(0,0)}(u)$  is a perfectly well defined function. But this difficulty has also appeared in our earlier research on multiple scattering, and we have been able to show in simpler cases that a completely correct but more complicated proof can be given, leading to the same results as the formal proof. Therefore, there is no doubt about the validity of (29) and the same kind of mathematical scheme may be used to transform the other integrals in (21).

First of all, we give a list of addition theorems involving the various elements of the triangle  $OAB$ . They can all be deduced from the first theorem by partial differentiation with respect to  $r$ ,  $r''$  or  $\xi$ :

$$(30) \quad \frac{\sin r'u}{r'u} = \sum_{k=0}^{\infty} (2k+1) P_k(\cos \xi) \frac{j_{k+\frac{1}{2}}(ru)}{ru} \cdot \frac{j_{k+\frac{1}{2}}(r''u)}{r''u},$$

$$(31) \quad \frac{j_{\frac{3}{2}}(r'u)}{r'u} \cos \theta = \sum_{k=0}^{\infty} (2k+1) P_k(\cos \xi) \frac{j_{k+\frac{1}{2}}(ru)}{ru} \cdot \left[ \frac{d}{dz} \frac{j_{k+\frac{1}{2}}(z)}{z} \right]_{z \equiv r''u},$$

$$(32) \quad \frac{j_{\frac{3}{2}}(r'u)}{r'u} \sin \theta = \sum_{k=1}^{\infty} (2k+1) P_k^1(\cos \xi) \frac{j_{k+\frac{1}{2}}(ru)}{ru} \cdot \frac{j_{k+\frac{1}{2}}(r''u)}{(r''u)^2},$$

$$(33) \quad \frac{j_{\frac{3}{2}}(r'u)}{r'u} \cos \beta = - \sum_{k=0}^{\infty} (2k+1) P_k(\cos \xi) \frac{j_{k+\frac{1}{2}}(r''u)}{r''u} \cdot \left[ \frac{d}{dz} \frac{j_{k+\frac{1}{2}}(z)}{z} \right]_{z \equiv ru},$$

$$(34) \quad \frac{j_{\frac{3}{2}}(r'u)}{r'u} \sin \beta = \sum_{k=1}^{\infty} (2k+1) P_k^1(\cos \xi) \frac{j_{k+\frac{1}{2}}(ru)}{(ru)^2} \cdot \frac{j_{k+\frac{1}{2}}(r''u)}{r''u},$$

$$(35) \quad \frac{j_{\frac{3}{2}}(r'u)}{r'u} P_2(\cos \theta) = \sum_{k=0}^{\infty} (2k+1) P_k(\cos \xi) \frac{j_{k+\frac{1}{2}}(ru)}{ru} \left[ i^2 P_2 \left( \frac{d}{i dz} \right) \frac{j_{k+\frac{1}{2}}(z)}{z} \right]_{z \equiv r''u},$$

$$(36) \quad \frac{j_{\frac{3}{2}}(r'u)}{r'u} \sin \theta \cos \theta = \sum_{k=1}^{\infty} (2k+1) P_k^1(\cos \xi) \frac{j_{k+\frac{1}{2}}(ru)}{ru} \left[ \frac{d}{dz} \frac{j_{k+\frac{1}{2}}(z)}{z^2} \right]_{z \equiv r''u}.$$

The formulae (30), (31) and (35) are special cases of the general addition

theorem which we have proved in a previous article <sup>(2)</sup>:

$$(37) \quad \frac{j_{m+\frac{1}{2}}(r'u)}{r'u} P_m(\cos \theta) = \\ = \sum_{k=0}^{\infty} (2k+1) P_k(\cos \xi) \frac{j_{k+\frac{1}{2}}(ru)}{ru} \left[ i^m P_m \left( \frac{d}{i dz} \right) \frac{j_{k+\frac{1}{2}}(z)}{z} \right]_{z=r'u}.$$

Adding to this the formal representation of  $(1/r)f(r)$  as a Fourier-Bessel integral, namely

$$(38) \quad \frac{f(r)}{r} = \frac{2}{\pi} \int_0^{\infty} u F^{(k,0)}(u) j_{k+\frac{1}{2}}(ru) du,$$

we can go over to the complete integration of (21).

1) Using (38) for  $k=0$  and applying (30), we have obtained (29).

2) Using (38) for  $k=1$  and applying (31), we get

$$(39) \quad \int_0^{\infty} \frac{f(r')f(r'')}{r'^2} \cos \theta dr'' = \\ = \frac{2}{\pi r} \sum_{k=0}^{\infty} (2k+1) P_k(\cos \xi) \int_0^{\infty} u F^{(1,0)}(u) F^{(k,1)}(u) j_{k+\frac{1}{2}}(ru) du.$$

3) Using (38) for  $k=1$  and applying (33), we find

$$(40) \quad \int_0^{\infty} \frac{f(r')f(r'')}{r'^2} \cos \beta dr'' = \\ = \frac{2}{\pi} \sum_{k=0}^{\infty} (2k+1) P_k(\cos \xi) \int_0^{\infty} u^2 F^{(1,0)}(u) F^{(k,0)}(u) \left[ -\frac{d}{dz} \frac{j_{k+\frac{1}{2}}(z)}{z} \right]_{z=ru} du.$$

Taking into account the recurrence relations between the Bessel functions, we can show that

$$(41) \quad \frac{d}{dz} \frac{j_{k+\frac{1}{2}}(z)}{z} = \frac{1}{(2k+1)z} [k j_{k-\frac{1}{2}}(z) - (k+1) j_{k+\frac{3}{2}}(z)],$$

<sup>(2)</sup> C. C. GROSJEAN: *Verh. Kon. Vl. Acad. Wetensch. (Belgium)*, **13**, n. 36 (1951).



so that (40) becomes

$$\begin{aligned}
 (42) \quad & \int_0^\infty \frac{f(r')f(r'')}{r'^2} \cos \beta \, dr'' = \\
 & = \frac{2}{\pi r} \left[ \sum_{k=0}^\infty (k+1) P_k(\cos \xi) \int_0^\infty u F^{(1,0)}(u) F^{(k,0)}(u) j_{k+\frac{3}{2}}(ru) \, du - \right. \\
 & \quad \left. - \sum_{k=1}^\infty k P_k(\cos \xi) \int_0^\infty u F^{(1,0)}(u) F^{(k,0)}(u) j_{k-\frac{3}{2}}(ru) \, du \right].
 \end{aligned}$$

4) Using (38) for  $k=1$  and applying (34), we get

$$\begin{aligned}
 (43) \quad & \int_0^\infty \frac{f(r')f(r'')}{r'^2} \sin \beta \, dr'' = \\
 & = \frac{2}{\pi r^2} \sum_{k=1}^\infty (2k+1) P_k^1(\cos \xi) \int_0^\infty F^{(1,0)}(u) F^{(k,0)}(u) j_{k+\frac{3}{2}}(ru) \, du.
 \end{aligned}$$

(41) was a first recurrence relation between the  $j$ -functions. The second relation is

$$(44) \quad \frac{j_{k+\frac{1}{2}}(z)}{z} = \frac{1}{(2k+1)} [j_{k+\frac{3}{2}}(z) + j_{k-\frac{1}{2}}(z)]$$

and we obtain for (43)

$$\begin{aligned}
 (45) \quad & \int_0^\infty \frac{f(r')f(r'')}{r'^2} \sin \beta \, dr'' = \\
 & = \frac{2}{\pi r} \sum_{k=1}^\infty P_k^1(\cos \xi) \left[ \int_0^\infty u F^{(1,0)}(u) F^{(k,0)}(u) j_{k+\frac{3}{2}}(ru) \, du + \right. \\
 & \quad \left. + \int_0^\infty u F^{(1,0)}(u) F^{(k,0)}(u) j_{k-\frac{3}{2}}(ru) \, du \right].
 \end{aligned}$$

5) The remaining integrals

$$(46) \quad \mathcal{J}_1 \equiv \int_0^\infty \frac{f(r')f(r'')}{r'^2} \cos \theta \cos \beta \, dr''; \quad \mathcal{J}_2 \equiv \int_0^\infty \frac{f(r')f(r'')}{r'^2} \cos \theta \sin \beta \, dr''$$

are more complicated to work out. Knowing that

$$(47) \quad \begin{cases} \cos \beta = \cos \theta \cos \xi + \sin \theta \sin \xi, \\ \sin \beta = \sin \theta \cos \xi - \cos \theta \sin \xi, \end{cases}$$

we may write

$$(48) \quad \mathcal{J}_1 = \frac{1}{3} \cos \xi \int_0^\infty \frac{f(r')f(r'')}{r'^2} dr'' + \frac{2}{3} \cos \xi \int_0^\infty \frac{f(r')f(r'')}{r'^2} P_2(\cos \theta) dr'' + \\ + \sin \xi \int_0^\infty \frac{f(r')f(r'')}{r'^2} \sin \theta \cos \theta dr'',$$

$$(49) \quad \mathcal{J}_2 = \cos \xi \int_0^\infty \frac{f(r')f(r'')}{r'^2} \sin \theta \cos \theta dr'' - \frac{1}{3} \sin \xi \int_0^\infty \frac{f(r')f(r'')}{r'^2} dr'' - \\ - \frac{2}{3} \sin \xi \int_0^\infty \frac{f(r')f(r'')}{r'^2} P_2(\cos \theta) dr''.$$

and now we can use the expansions (35) and (36):

$$(50) \quad \mathcal{J}_1 = \frac{2 \cos \xi}{3\pi r} \sum_{k=0}^\infty (2k+1) P_k(\cos \xi) \int_0^\infty u F^{(0,0)}(u) F^{(k,0)}(u) j_{k+\frac{1}{2}}(ru) du + \\ + \frac{4 \cos \xi}{3\pi r} \sum_{k=0}^\infty (2k+1) P_k(\cos \xi) \int_0^\infty u F^{(2,0)}(u) F^{(k,2)}(u) j_{k+\frac{1}{2}}(ru) du + \\ + \frac{2 \sin \xi}{\pi r} \sum_{k=1}^\infty (2k+1) P_k^1(\cos \xi) \int_0^\infty u F^{(2,0)}(u) j_{k+\frac{1}{2}}(ru) du \int_0^\infty f(r'') \left[ \frac{d}{dz} \frac{j_{k+\frac{1}{2}}(z)}{z^2} \right]_{z=r''u} dr''.$$

Taking into account the relations:

$$F^{(k,2)}(u) = \frac{1}{2} F^{(k,0)}(u) + \frac{3}{2} \int_0^\infty f(r'') \left[ \frac{d^2}{dz^2} \frac{j_{k+\frac{1}{2}}(z)}{z} \right]_{z=r''u} dr'' = \\ = \frac{1}{2} F^{(k,0)}(u) + \frac{3}{2(2k+1)} [k F^{(k-1,1)}(u) - (k+1) F^{(k+1,1)}(u)],$$

$$\int_0^{\infty} f(r'') \left[ \frac{d}{dz} \frac{j_{k+\frac{1}{2}}(z)}{z^2} \right]_{z=r''u} dr'' = \frac{1}{(2k+1)} \left[ F^{(k-1,1)}(u) + F^{(k+1,1)}(u) \right],$$

$$\sin \xi P_k^1(\cos \xi) = k P_{k-1}(\cos \xi) - k \cos \xi P_k(\cos \xi),$$

$$(2k+1) \cos \xi P_k(\cos \xi) = k P_{k-1}(\cos \xi) + (k+1) P_{k+1}(\cos \xi),$$

we find:

$$\begin{aligned} (51) \quad \mathcal{J}_1 &= \frac{2}{3\pi r} \sum_{k=1}^{\infty} k P_{k-1}(\cos \xi) \int_0^{\infty} u [F^{(0,0)}(u) + F^{(2,0)}(u)] F^{(k,0)}(u) j_{k+\frac{1}{2}}(ru) du + \\ &+ \frac{2}{3\pi r} \sum_{k=0}^{\infty} (k+1) P_{k+1}(\cos \xi) \int_0^{\infty} u [F^{(0,0)}(u) + F^{(2,0)}(u)] F^{(k,0)}(u) j_{k+\frac{1}{2}}(ru) du + \\ &+ \frac{2 \cos \xi}{\pi r} \sum_{k=0}^{\infty} P_k(\cos \xi) \cdot \\ &\quad \cdot \int_0^{\infty} u F^{(2,0)}(u) [k F^{(k-1,1)}(u) - (k+1) F^{(k+1,1)}(u)] j_{k+\frac{1}{2}}(ru) du + \\ &+ \frac{2}{\pi r} \sum_{k=1}^{\infty} [k P_{k-1}(\cos \xi) - k \cos \xi P_k(\cos \xi)] \cdot \\ &\quad \cdot \int_0^{\infty} u F^{(2,0)}(u) [F^{(k-1,1)}(u) + F^{(k+1,1)}(u)] j_{k+\frac{1}{2}}(u) du = \\ &= \frac{2}{3\pi r} \sum_{k=0}^{\infty} (k+1) P_k(\cos \xi) \int_0^{\infty} u [F^{(0,0)}(u) - F^{(2,0)}(u)] F^{(k+1,0)}(u) j_{k+\frac{3}{2}}(ru) du + \\ &+ \frac{2}{3\pi r} \sum_{k=1}^{\infty} k P_k(\cos \xi) \int_0^{\infty} u [F^{(0,0)}(u) + F^{(2,0)}(u)] F^{(k-1,0)}(u) j_{k-\frac{1}{2}}(ru) du + \\ &+ \frac{2}{\pi r} \sum_{k=0}^{\infty} (k+1) P_k(\cos \xi) \int_0^{\infty} u F^{(2,0)}(u) F^{(k,1)}(u) j_{k+\frac{3}{2}}(ru) du - \\ &- \frac{2}{\pi r} \sum_{k=1}^{\infty} k P_k(\cos \xi) \int_0^{\infty} u F^{(2,0)}(u) F^{(k,1)}(u) j_{k-\frac{1}{2}}(ru) du. \end{aligned}$$

The integral  $\mathcal{J}_2$  can be calculated in a similar way:

$$(52) \quad \mathcal{J}_2 = \frac{2 \cos \xi}{\pi r} \sum_{k=1}^{\infty} (2k+1) P_k^1(\cos \xi) \cdot$$

$$\cdot \int_0^{\infty} u F^{(2,0)}(u) j_{k+\frac{1}{2}}(ru) du \int_0^{\infty} f(r'') \left[ \frac{d}{dz} \frac{j_{k+\frac{1}{2}}(z)}{z^2} \right]_{z=r''u} dr'' -$$



$$\begin{aligned}
& - \frac{2 \sin \xi}{3\pi r} \sum_{k=0}^{\infty} (2k+1) P_k(\cos \xi) \int_0^{\infty} u F^{(0,0)}(u) F^{(k,0)}(u) j_{k+\frac{1}{2}}(ru) du - \\
& - \frac{4 \sin \xi}{3\pi r} \sum_{k=0}^{\infty} (2k+1) P_k(\cos \xi) \int_0^{\infty} u F^{(2,0)}(u) F^{(k,2)}(u) j_{k+\frac{1}{2}}(ru) du = \\
& = \frac{2 \sin \xi}{\pi r} \sum_{k=1}^{\infty} \cos \xi P'_k(\cos \xi) \int_0^{\infty} u F^{(2,0)}(u) [F^{(k-1,1)}(u) + F^{(k+1,1)}(u)] j_{k+\frac{1}{2}}(ru) du - \\
& - \frac{2 \sin \xi}{3\pi r} \sum_{k=0}^{\infty} (2k+1) P_k(\cos \xi) \int_0^{\infty} u [F^{(0,0)}(u) + F^{(2,0)}(u)] F^{(k,0)}(u) j_{k+\frac{1}{2}}(ru) du - \\
& - \frac{2 \sin \xi}{\pi r} \sum_{k=0}^{\infty} P_k(\cos \xi) \int_0^{\infty} u F^{(2,0)}(u) [k F^{(k-1,1)}(u) - (k+1) F^{(k+1,1)}(u)] j_{k+\frac{1}{2}}(ru) du
\end{aligned}$$

and using the recurrence relations

$$\begin{aligned}
(2k+1) P_k(\cos \xi) &= P'_{k+1}(\cos \xi) - P'_{k-1}(\cos \xi) \\
P'_k(\cos \xi) \cos \xi - k P_k(\cos \xi) &= P'_{k-1}(\cos \xi) \\
P'_k(\cos \xi) \cos \xi + (k+1) P_k(\cos \xi) &= P'_{k+1}(\cos \xi),
\end{aligned}$$

we obtain as final result:

$$\begin{aligned}
(53) \quad \mathcal{J}_2 &= \frac{2}{\pi r} \sum_{k=1}^{\infty} P_k^1(\cos \xi) \int_0^{\infty} u F^{(2,0)}(u) F^{(k,1)}(u) j_{k+\frac{3}{2}}(ru) du + \\
& + \frac{2}{\pi r} \sum_{k=1}^{\infty} P_k^1(\cos \xi) \int_0^{\infty} u F^{(2,0)}(u) F^{(k,1)}(u) j_{k-\frac{3}{2}}(u) du + \\
& + \frac{2}{3\pi r} \sum_{k=1}^{\infty} P_k^1(\cos \xi) \int_0^{\infty} u [F^{(0,0)}(u) + F^{(2,0)}(u)] F^{(k+1,0)}(u) j_{k+\frac{3}{2}}(ru) du - \\
& - \frac{2}{3\pi r} \sum_{k=1}^{\infty} P_k^1(\cos \xi) \int_0^{\infty} u [F^{(0,0)}(u) + F^{(2,0)}(u)] F^{(k-1,0)}(u) j_{k-\frac{3}{2}}(ru) du.
\end{aligned}$$

Putting all these results into (21), the final form of  $\varphi_2$  can be represented

as follows:

$$\begin{aligned}
 (54) \quad \varphi_2(r, \alpha; \xi, \nu) = & r \sum_{k=0}^{\infty} (2k+1) P_k(\cos \xi) \int_0^{\infty} g_{2,0}^{(k,0)}(u) j_{k+\frac{1}{2}}(ru) du + \\
 & + r \cos \alpha \left[ \sum_{k=0}^{\infty} (k+1) P_k(\cos \xi) \int_0^{\infty} g_{2,1}^{(k,0)}(u) j_{k+\frac{3}{2}}(ru) du - \right. \\
 & \left. - \sum_{k=1}^{\infty} k P_k(\cos \xi) \int_0^{\infty} h_{2,1}^{(k,0)}(u) j_{k-\frac{1}{2}}(ru) du \right] + \\
 & + r \sin \alpha \cos \nu \sum_{k=1}^{\infty} P_k^1(\cos \xi) \left[ \int_0^{\infty} g_{2,1}^{(k,0)}(u) j_{k+\frac{3}{2}}(ru) du + \right. \\
 & \left. + \int_0^{\infty} h_{2,1}^{(k,0)}(u) j_{k-\frac{1}{2}}(ru) du \right]
 \end{aligned}$$

in which

$$(55) \quad \left\{ \begin{aligned} g_{2,0}^{(k,0)}(u) &= \frac{2u}{\pi} [F^{(0,0)}(u) F^{(k,0)}(u) + 3b F^{(1,0)}(u) F^{(k,1)}(u)], \\ g_{2,1}^{(k,0)}(u) &= \frac{2Au}{\pi} [F^{(1,0)}(u) F^{(k,0)}(u) + 3b F^{(2,0)}(u) F^{(k,1)}(u) + \\ &\quad + b(F^{(0,0)}(u) + F^{(2,0)}(u)) F^{(k+1,0)}(u)], \\ h_{2,1}^{(k,0)}(u) &= \frac{2Au}{\pi} [F^{(1,0)}(u) F^{(k,0)}(u) + 3b F^{(2,0)}(u) F^{(k,1)}(u) - \\ &\quad - b(F^{(0,0)}(u) + F^{(2,0)}(u)) F^{(k-1,0)}(u)]. \end{aligned} \right.$$

**Solution of  $\varphi_n(r, \alpha; \xi, \nu)$ .**

The most logical way to construct the exact mathematical form of  $\varphi_n$  consists in postulating the following expression, suggested by the result for  $\varphi_2$ :

$$\begin{aligned}
 (56) \quad \varphi_n(r, \alpha; \xi, \nu) = & r \sum_{k=0}^{\infty} (2k+1) P_k(\cos \xi) \int_0^{\infty} g_{n,0}^{(k,0)}(u) j_{k+\frac{1}{2}}(ru) du + \\
 & + r \cos \alpha \left[ \sum_{k=0}^{\infty} (k+1) P_k(\cos \xi) \int_0^{\infty} g_{n,1}^{(k,0)}(u) j_{k+\frac{3}{2}}(ru) du - \right.
 \end{aligned}$$

$$\begin{aligned}
& - \sum_{k=1}^{\infty} k P_k(\cos \xi) \int_0^{\infty} h_{n,1}^{(k,0)}(u) j_{k-\frac{1}{2}}(ru) du \Big| + \\
& + r \sin \alpha \cos \nu \sum_{k=1}^{\infty} P_k^1(\cos \xi) \left[ \int_0^{\infty} g_{n,1}^{(k,0)}(u) j_{k+\frac{3}{2}}(ru) du + \right. \\
& \left. + \int_0^{\infty} h_{n,1}^{(k,0)}(u) j_{k-\frac{1}{2}}(ru) du \right] \quad (n = 2, 3, \dots).
\end{aligned}$$

To prove this, we have to replace the components of  $\varphi_{n-1}$  in (17) by integrals deduced from (56) and calculate  $\varphi_n$  in order to compare it with the proposed form of the solution. Based on (56), we may write

$$\begin{aligned}
(57) \quad \varphi_{n-1}(r', \alpha'; \theta, \zeta) = & \sum_{k=0}^{\infty} [\varphi_{n-1,0}^{(k,0)}(r') + \varphi_{n-1,1}^{(k,0)}(r') \cos \alpha'] P_k(\cos \theta) + \\
& + \sin \alpha' \cos \zeta \sum_{k=1}^{\infty} \varphi_{n-1}^{(k,1)}(r') P_k^1(\cos \theta)
\end{aligned}$$

and (17) becomes:

$$\begin{aligned}
(58) \quad \varphi_n(r, \alpha; \xi, \nu) = & r^2 \int_0^{\infty} \frac{\varphi_{n-1,0}^{(0,0)}(r') + \varphi_{n-1,1}^{(0,0)}(r') \cos \alpha'}{r'^2} f(r'') dr'' + \\
& + br^2 \int_0^{\infty} \frac{\varphi_{n-1,0}^{(1,0)}(r') + \varphi_{n-1,1}^{(1,0)}(r') \cos \alpha'}{r'^2} f(r'') \cos \theta dr'' + \\
& + br^2 \int_0^{\infty} \frac{\varphi_{n-1}^{(1,1)}(r')}{r'^2} f(r'') \sin \alpha' \sin \theta \cos \zeta dr''.
\end{aligned}$$

Considering the trihedral  $OzAB$  (Fig. 2), we have

$$\cos \alpha' = \cos \alpha \cos \beta + \sin \alpha \sin \beta \cos \nu$$

and

$$\cos \alpha = \cos \alpha' \cos \beta - \sin \alpha' \sin \beta \cos \zeta$$

so that

$$\sin \alpha' \cos \zeta = \sin \alpha \cos \beta \cos \nu - \cos \alpha \sin \beta$$

and (58) is transformed into:

$$(59) \quad \varphi_n(r, \alpha; \xi, \nu) = r^2 \int_0^{\infty} \frac{\varphi_{n-1,0}^{(0,0)}(r')}{r'^2} f(r'') dr'' + br^2 \int_0^{\infty} \frac{\varphi_{n-1,0}^{(1,0)}(r')}{r'^2} f(r'') \cos \theta dr'' +$$



$$\begin{aligned}
& + \cos \alpha \left[ r^2 \int_0^\infty \frac{\varphi_{n-1,1}^{(0,0)}(r')}{r'^2} f(r'') \cos \beta \, dr'' + br^2 \int_0^\infty \frac{\varphi_{n-1,1}^{(1,0)}(r')}{r'^2} f(r'') \cos \theta \cos \beta \, dr'' - \right. \\
& \left. - br^2 \int_0^\infty \frac{\varphi_{n-1,1}^{(1,1)}(r')}{r'^2} f(r'') \sin \theta \sin \beta \, dr'' \right] + \\
& + \sin \alpha \cos \nu \left[ r^2 \int_0^\infty \frac{\varphi_{n-1,1}^{(0,0)}(r')}{r'^2} f(r'') \sin \beta \, dr'' + br^2 \int_0^\infty \frac{\varphi_{n-1,1}^{(1,0)}(r')}{r'^2} f(r'') \cos \theta \sin \beta \, dr'' + \right. \\
& \left. + br^2 \int_0^\infty \frac{\varphi_{n-1,1}^{(1,1)}(r')}{r'^2} f(r'') \sin \theta \cos \beta \, dr'' \right].
\end{aligned}$$

Now, we can make use of the same addition theorems and the same recurrence relations as for the calculation of  $\varphi_2$ . The result is that  $\varphi_n$  is precisely of the form (56) which we expected, and in addition to this, we obtain the following recurrence relations:

$$(60) \quad \left\{ \begin{aligned} g_{n,0}^{(k,0)}(u) &= g_{n-1,0}^{(0,0)}(u) F^{(k,0)}(u) + 3b_{n-1,0}'^{(1,0)}(u) F^{(k,1)}(u) & (k=0, 1, \dots), \\ g_{n,1}^{(k,0)}(u) &= g_{n-1,1}^{(0,0)}(u) F^{(k,0)}(u) + 3b_{n-1,1}^{(1,0)}(u) F^{(k,1)}(u) + \\ &+ b[g_{n-1,1}^{(1,0)}(u) - h_{n-1,1}^{(1,0)}(u)] F^{(k+1,0)}(u) & (k=0, 1, \dots), \\ g_{n,1}^{(k,0)}(u) &= g_{n-1,1}^{(0,0)}(u) F^{(k,0)}(u) + 3bg_{n-1,1}^{(1,0)}(u) F^{(k,1)}(u) - \\ &- b[g_{n-1,1}^{(1,0)}(u) - h_{n-1,1}^{(1,0)}(u)] F^{(k-1,0)}(u) & (k=1, 2, \dots) \quad (n=3, 4, \dots). \end{aligned} \right.$$

Starting with (55), it is theoretically possible to calculate the  $g$  and the  $h$ -functions for  $n=3, 4$ , etc., leading to the exact mathematical expressions of  $\varphi_3, \varphi_4$ , etc.. However, as we have stated in the beginning of this article, we are particularly interested in the total distributions (7).

### Calculation of $\Phi(r, \alpha; \xi, \nu)$ .

Although the relations (60) are so far only valid for  $n \geq 3$ , we can write them out for  $n=2$  and make a comparison with (55). In this way, we have introduced some new  $g$  and  $h$ -functions, formally related to the first collision.

Indeed, if we put

$$(61) \quad \left\{ \begin{array}{l} g_{1,0}^{(0,0)}(u) = \frac{2u}{\pi} F^{(0,0)}(u); \quad g_{1,0}^{(1,0)}(u) = \frac{2u}{\pi} F^{(1,0)}(u); \\ g_{1,1}^{(0,0)}(u) = \frac{2Au}{\pi} F^{(1,0)}(u); \quad g_{1,1}^{(1,0)}(u) = \frac{2Au}{\pi} F^{(2,0)}(u); \\ h_{1,1}^{(1,0)}(u) = -\frac{2Au}{\pi} F^{(0,0)}(u), \end{array} \right.$$

the relations (60) become valid for  $n \geq 2$  and they include (55). That the functions in (61) are formally related to  $n = 1$ , can be seen as follows. Suppose that we would try to represent  $\varphi_1(r, \alpha; \xi, \nu)$  formally by (56) in which we put  $n = 1$ . Using (8) and expanding  $\delta(\xi)$ , we could write

$$(62) \quad \varphi_1(r, \alpha; \xi, \nu) = f(r)(1 + A \cos \alpha) \sum_{k=0}^{\infty} (2k+1) P_k(\cos \xi)$$

and comparing with (56), we must have

$$(63) \quad \left\{ \begin{array}{l} r \int_0^{\infty} g_{1,0}^{(k,0)}(u) j_{k+\frac{1}{2}}(ru) du = f(r) \quad (k = 0, 1, \dots), \\ r \left[ (k+1) \int_0^{\infty} g_{1,1}^{(k,0)}(u) j_{k+\frac{3}{2}}(ru) du - k \int_0^{\infty} h_{1,1}^{(k,0)}(u) j_{k-\frac{1}{2}}(ru) du \right] = \\ \quad \quad \quad = A(2k+1)f(r) \quad (k = 0, 1, \dots), \\ \int_0^{\infty} g_{1,1}^{(k,0)}(u) j_{k+\frac{3}{2}}(ru) du + \int_0^{\infty} h_{1,1}^{(k,0)}(u) j_{k-\frac{1}{2}}(ru) du = 0 \quad (k = 1, 2, \dots). \end{array} \right.$$

The solution of this set of integral equations is:

$$(64) \quad \left\{ \begin{array}{l} g_{1,0}^{(k,0)}(u) = \frac{2u}{\pi} F^{(k,0)}(u) \quad (k = 0, 1, \dots), \\ g_{1,1}^{(k,0)}(u) = \frac{2Au}{\pi} F^{(k+1,0)}(u) \quad (k = 0, 1, \dots), \\ h_{1,1}^{(k,0)}(u) = -\frac{2Au}{\pi} F^{(k-1,0)}(u) \quad (k = 1, 2, \dots). \end{array} \right.$$

The results (61) are special cases of the more general formulae (64).

Now, we can finally calculate the total distribution  $\Phi(r, \alpha; \xi, \nu)$ . Defining the symbols

$$(65) \quad g_0^{(k,0)}(u) \equiv \sum_{n=1}^{\infty} g_{n,0}^{(k,0)}(u); \quad g_1^{(k,0)}(u) \equiv \sum_{n=1}^{\infty} g_{n,1}^{(k,0)}(u); \quad h_1^{(k,0)}(u) \equiv \sum_{n=1}^{\infty} h_{n,1}^{(k,0)}(u);$$

$$(66) \quad G_0^{(k)}(u) \equiv \sum_{n=2}^{\infty} g_{n,0}^{(k,0)}(u); \quad G_1^{(k)}(u) \equiv \sum_{n=2}^{\infty} g_{n,1}^{(k,0)}(u); \quad H_1^{(k)}(u) \equiv \sum_{n=2}^{\infty} h_{n,1}^{(k,0)}(u),$$

and summing up with respect to  $n$  in (56), we obtain the exact mathematical form of  $\Phi(r, \alpha; \xi, \nu)$ :

$$(67) \quad \begin{aligned} \Phi(r, \alpha; \xi, \nu) = & \frac{2\delta(\xi)f(r)(1 + A \cos \alpha)}{\sin \xi} + r \sum_{k=0}^{\infty} (2k+1)P_k(\cos \xi) \int_0^{\infty} G_0^{(k)}(u) j_{k+\frac{1}{2}}(ru) du + \\ & + r \cos \alpha \left[ \sum_{k=0}^{\infty} (k+1)P_k(\cos \xi) \int_0^{\infty} G_1^{(k)}(u) j_{k+\frac{3}{2}}(ru) du - \right. \\ & \left. - \sum_{k=1}^{\infty} kP_k(\cos \xi) \int_0^{\infty} H_1^{(k)}(u) j_{k-\frac{1}{2}}(ru) du \right] + \\ & + r \sin \alpha \cos \nu \sum_{k=1}^{\infty} P_k^1(\cos \xi) \left[ \int_0^{\infty} G_1^{(k)}(u) j_{k+\frac{3}{2}}(ru) du + \int_0^{\infty} H_1^{(k)}(u) j_{k-\frac{1}{2}}(ru) du \right]. \end{aligned}$$

By summation with respect to  $n$  in (60), we get a system of linear equations for the  $g$  and the  $h$ -functions:

$$(68) \quad g_0^{(k,0)}(u) - g_{1,0}^{(k,0)}(u) = g_0^{(0,0)}(u)F^{(k,0)}(u) + 3bg_0^{(1,0)}(u)F^{(k,1)}(u) \quad (k=0, 1, \dots),$$

$$(69) \quad \begin{aligned} g_1^{(k,0)}(u) - g_{1,1}^{(k,0)}(u) = & g_1^{(0,0)}(u)F^{(k,0)}(u) + 3bg_1^{(1,0)}(u)F^{(k,1)}(u) + \\ & + b[g_1^{(1,0)}(u) - h_1^{(1,0)}(u)]F^{(k+1,0)}(u) \quad (k=0, 1, \dots), \end{aligned}$$

$$(70) \quad \begin{aligned} h_1^{(k,0)}(u) - h_{1,1}^{(k,0)}(u) = & g_1^{(0,0)}(u)F^{(k,0)}(u) + 3bg_1^{(1,0)}(u)F^{(k,1)}(u) - \\ & - b[g_1^{(1,0)}(u) - h_1^{(1,0)}(u)]F^{(k-1,0)}(u) \quad (k=1, 2, \dots). \end{aligned}$$



Applying (68) for  $k = 0$  and  $k = 1$ , and taking into account (61), we find the following system:

$$(71) \quad \begin{cases} [1 - F^{(0,0)}(u)]g_0^{(0,0)}(u) - 3bF^{(0,1)}(u)g_0^{(1,0)}(u) = \frac{2u}{\pi} F^{(0,0)}(u) \\ -F^{(1,0)}(u)g_0^{(0,0)}(u) + [1 - 3bF^{(1,1)}(u)]g_0^{(1,0)}(u) = \frac{2u}{\pi} F^{(1,0)}(u) . \end{cases}$$

The solution is

$$(72) \quad \left\{ \begin{aligned} g_0^{(0,0)}(u) &= \frac{2u}{\pi} \cdot \\ &\cdot \left\{ 1 - F^{(0,0)}(u) - 3bF^{(1,1)}(u) + 3b \frac{F^{(0,0)}(u) - 3b[F^{(0,0)}(u)F^{(1,1)}(u) - F^{(1,0)}(u)F^{(0,1)}(u)]}{F^{(0,0)}(u)F^{(1,1)}(u) - F^{(1,0)}(u)F^{(0,1)}(u)} \right\} \\ g_0^{(1,0)}(u) &= \frac{2u}{\pi} \cdot \\ &\cdot \frac{F^{(1,0)}(u)}{1 - F^{(0,0)}(u) - 3bF^{(1,1)}(u) + 3b[F^{(0,0)}(u)F^{(1,1)}(u) - F^{(1,0)}(u)F^{(0,1)}(u)]} . \end{aligned} \right.$$

This solution can be introduced in (68) and we get the explicit form of the first set of  $G$ -functions:

$$(73) \quad G_0^{(k)}(u) = \frac{2u}{\pi} \cdot \frac{F^{(k,0)}[F^{(0,0)} - 3b(F^{(0,0)}F^{(1,1)} - F^{(1,0)}F^{(0,1)})] + 3bF^{(k,1)}F^{(1,0)}}{[1 - F^{(0,0)} - 3bF^{(1,1)} + 3b(F^{(0,0)}F^{(1,1)} - F^{(1,0)}F^{(0,1)})]} ,$$

in which we have replaced  $F^{(k,m)}(u)$  by  $F^{(k,m)}$  for brevity's sake.

In order to solve the simultaneous equations (69) and (70), we can first of all subtract them:

$$(74) \quad g_1^{(k,0)}(u) - h_1^{(k,0)}(u) - \frac{2Au}{\pi} [F^{(k+1,0)}(u) + F^{(k-1,0)}(u)] = \\ = b[g_1^{(1,0)}(u) - h_1^{(1,0)}(u)][F^{(k+1,0)}(u) + F^{(k-1,0)}(u)] .$$

For  $k = 1$ , we get

$$(75) \quad g_1^{(1,0)}(u) - h_1^{(1,0)}(u) = \frac{2Au}{\pi} \cdot \frac{F^{(2,0)} + F^{(0,0)}}{[1 - b(F^{(2,0)} + F^{(0,0)})]}$$

and hence for all  $k \geq 1$ :

$$(76) \quad g_1^{(k,0)}(u) - h_1^{(k,0)}(u) = \frac{2Au}{\pi} \cdot \frac{F^{(k+1,0)} + F^{(k-1,0)}}{[1 - b(F^{(0,0)} + F^{(2,0)})]} \quad (k = 1, 2, \dots) .$$

Using (75) in (69) and writing the equations for  $k = 0$  and  $k = 1$ , we find:

$$(77) \quad \left\{ \begin{aligned} [1 - F^{(0,0)}(u)]g_1^{(0,0)}(u) - 3bF^{(0,1)}(u)g_1^{(1,0)}(u) &= \\ &= \frac{2Au}{\pi} \cdot \frac{F^{(1,0)}(u)}{\{1 - b[F^{(0,0)}(u) + F^{(2,0)}(u)]\}} \\ - F^{(1,0)}(u)g_1^{(0,0)}(u) + [1 - 3bF^{(1,1)}(u)]g_1^{(1,0)}(u) &= \\ &= \frac{2Au}{\pi} \cdot \frac{F^{(2,0)}(u)}{\{1 - b[F^{(0,0)}(u) + F^{(2,0)}(u)]\}} \end{aligned} \right.$$

with the following solution:

$$(78) \quad \left\{ \begin{aligned} g_1^{(0,0)}(u) &= \frac{2Au}{\pi} \cdot \frac{F^{(1,0)}(u)}{[1 - F^{(0,0)} - 3bF^{(1,1)} + 3b(F^{(0,0)}F^{(1,1)} - F^{(1,0)}F^{(0,1)})]} , \\ g_1^{(1,0)}(u) &= \frac{2Au}{\pi} \cdot \\ &\cdot \frac{F^{(2,0)}(1 - F^{(0,0)}) + [F^{(1,0)}]^2}{[1 - b(F^{(0,0)} + F^{(2,0)})][1 - F^{(0,0)} - 3bF^{(1,1)} + 3b(F^{(0,0)}F^{(1,1)} - F^{(1,0)}F^{(0,1)})]} = \\ &= \frac{2Au}{3b\pi} \left[ \frac{1}{[1 - b(F^{(0,0)} + F^{(2,0)})]} \right. \\ &\quad \left. \frac{1 - F^{(0,0)}(u)}{[1 - F^{(0,0)} - 3bF^{(1,1)} + 3b(F^{(0,0)}F^{(1,1)} - F^{(1,0)}F^{(0,1)})]} \right] . \end{aligned} \right.$$

Now we put these results into (69) and (70), and we obtain:

$$(79) \quad G_1^{(k)}(u) = \frac{2Au}{\pi} \cdot \frac{F^{(k,0)}F^{(1,0)} - F^{(k,1)}(1 - F^{(0,0)})}{[1 - F^{(0,0)} - 3bF^{(1,1)} + 3b(F^{(0,0)}F^{(1,1)} - F^{(1,0)}F^{(0,1)})]} + \\ + \frac{2Au}{\pi} \cdot \frac{F^{(k,1)} + bF^{(k+1,0)}(F^{(0,0)} + F^{(2,0)})}{[1 - b(F^{(0,0)} + F^{(2,0)})]} ,$$

$$(80) \quad H_1^{(k)}(u) = \frac{2Au}{\pi} \cdot \frac{F^{(k,0)}F^{(1,0)} - F^{(k,1)}(1 - F^{(0,0)})}{[1 - F^{(0,0)} - 3bF^{(1,1)} + 3b(F^{(0,0)}F^{(1,1)} - F^{(1,0)}F^{(0,1)})]} + \\ + \frac{2Au}{\pi} \cdot \frac{F^{(k,1)} - bF^{(k-1,0)}(F^{(0,0)} + F^{(2,0)})}{[1 - b(F^{(0,0)} + F^{(2,0)})]} .$$

The final result of our calculations is the exact formula for  $\Phi(r, \alpha; \xi, \nu)$  represented by (67), in which we have to replace the  $G$  and the  $H$ -functions given by (73), (79) and (80).

### Application to thermal Neutrons.

In this paragraph, we wish to find the thermal neutron density and current in every point of space, when the source strength is described by

$$(81) \quad (S_0 + S_1 \cos \alpha) \frac{\sin \alpha d\alpha}{2}.$$

Taking (5) as the probability distribution of the scattering angles, we shall obtain results, which are correct up to the first order of non-isotropy of the scattering.

Considering the definition of  $\Phi(r, \alpha; \xi, \nu)$ , we see that the density is given by

$$(82) \quad \varrho(r, \alpha) = \frac{S_0 \lambda_s}{4\pi \bar{v} r^2} \int_0^{2\pi} \frac{d\nu}{2\pi} \int_0^\pi \Phi(r, \alpha; \xi, \nu) \frac{\sin \xi d\xi}{2},$$

whereas the neutron current  $i(r, \alpha)$  is a vector having two componets  $i_z(r, \alpha)$  and  $i_x(r, \alpha)$  along the local  $AZ$  and  $AX$  axes in each point  $A$  of space:

$$(83) \quad i_z(r, \alpha) = \frac{S_0 \lambda_s}{4\pi r^2} \int_0^{2\pi} \frac{d\nu}{2\pi} \int_0^\pi \Phi(r, \alpha; \xi, \nu) \cos \xi \frac{\sin \xi d\xi}{2}$$

$$(84) \quad i_x(r, \alpha) = \frac{S_0 \lambda_s}{4\pi r^2} \int_0^{2\pi} \cos \nu \frac{d\nu}{2\pi} \int_0^\pi \Phi(r, \alpha; \xi, \nu) \frac{\sin^2 \xi d\xi}{2},$$

in which  $A \equiv S_1/S_0$  and

$\bar{v}$  is the average thermal neutron velocity.

Applying (67) and keeping in mind that in the present case the function  $f(l)$  is of the form (3), we find:

$$(85) \quad \varrho(r, \alpha) = \frac{S_0 \lambda_s}{4\pi \bar{v} r^2} \left[ \frac{e^{-r/\lambda}}{\lambda_s} + r \int_0^\infty G_0^{(0)}(u) \sin ru du \right] + \\ + \frac{S_1 \lambda_s \cos \alpha}{4\pi \bar{v} r^2} \left[ \frac{e^{-r/\lambda}}{\lambda_s} + \frac{r}{A} \int_0^\infty G_1^{(0)}(u) j_{\frac{3}{2}}(ru) du \right],$$

$$(86) \quad i_z(r, \alpha) = \frac{S_0 \lambda_s}{4\pi r^2} \left[ \frac{e^{-r/\lambda}}{\lambda_s} + r \int_0^\infty G_0^{(1)}(u) j_{\frac{3}{2}}(ru) du \right] +$$



$$\begin{aligned}
 & + \frac{S_1 \lambda_s \cos \alpha}{4\pi r^2} \left[ \frac{e^{-r/\lambda}}{\lambda_s} + \frac{2r}{3A} \int_0^\infty G_1^{(1)}(u) j_{\frac{3}{2}}(ru) du - \frac{r}{3A} \int_0^\infty H_1^{(1)}(u) \sin ru du \right], \\
 (87) \quad i_x(r, \alpha) &= \frac{S_1 \lambda_s \sin \alpha}{4\pi r} \cdot \frac{1}{3A} \left[ \int_0^\infty G_1^{(1)}(u) j_{\frac{3}{2}}(ru) du + \int_0^\infty H_1^{(1)}(u) \sin ru du \right].
 \end{aligned}$$

Now, let us calculate the  $G$  and the  $H$ -functions. Knowing that

$$\begin{aligned}
 (88) \quad F^{(k,0)}(u) &= \frac{1}{\lambda_s u} \int_0^\infty \exp \left[ -\frac{y}{\lambda} \right] j_{k+\frac{1}{2}}(yu) \frac{dy}{y} = \\
 &= \frac{1}{2\lambda_s i^k} \int_0^\infty \exp \left[ -\frac{y}{\lambda} \right] dy \int_0^\pi \exp [iyu \cos \theta] P_k(\cos \theta) \sin \theta d\theta = \\
 &= \frac{1}{2\lambda_s i^k} \int_0^\pi \frac{P_k(\cos \theta)}{((1/\lambda) - iu \cos \theta)} \sin \theta d\theta = \frac{1}{\lambda_s u i^{k+1}} Q_k \left( \frac{-i}{\lambda u} \right),
 \end{aligned}$$

$$\begin{aligned}
 (89) \quad F^{(k,1)}(u) &= \frac{1}{\lambda_s} \int_0^\infty \exp \left[ -\frac{y}{\lambda} \right] \left[ \frac{d}{dz} j_{k+\frac{1}{2}}(z) \right]_{z=yu} dy = \\
 &= \frac{1}{\lambda_s u} \left[ \frac{1}{\lambda u i^{k+1}} Q_k \left( \frac{-i}{\lambda u} \right) - \delta_{k0} \right],
 \end{aligned}$$

in which  $Q_k(\dots)$  represents the  $k$ -th order Legendre function of the second kind, we get:

$$(90) \quad \begin{cases} F^{(0,0)}(u) = \frac{\text{Arctg } \lambda u}{\lambda_s u}; & F^{(1,0)}(u) = -F^{(0,1)}(u) = \frac{1}{\lambda_s u} \left( 1 - \frac{\text{Arctg } \lambda u}{\lambda u} \right); \\ F^{(1,1)}(u) = \frac{1}{\lambda u} F^{(1,0)}(u); & F^{(2,0)}(u) = \frac{1}{2\lambda_s u} \left[ 3 \left( \frac{\text{Arctg } \lambda u}{\lambda u} - 1 \right) + \text{Arctg } \lambda u \right]. \end{cases}$$

These functions must be introduced into (73), (79) and (80). In this way, we find for example for  $G_0^{(0)}(u)$ :

$$(91) \quad G_0^{(0)}(u) = \frac{2}{\pi \lambda_s} \cdot \frac{(\text{Arctg } \lambda u)^2 + 3b \left( 1 - \frac{\text{Arctg } \lambda u}{\lambda u} \right) \left( \frac{\text{Arctg } \lambda u}{\lambda u} - 1 \right)}{\lambda_s u - \text{Arctg } \lambda u - \frac{3b}{\lambda_s u} \left( 1 - \frac{\text{Arctg } \lambda u}{\lambda u} \right)}.$$

With the obtained  $G$  and  $H$ -functions, we have to calculate the integrals in (85), (86) and (87). Unfortunately, they cannot be worked out in an exact closed form, but it is possible to calculate them by numerical integration using for instance punch card machines. In our earlier research on multiple scattering, we have computed a number of Fourier integrals in this way, using a 602 A multiplier of the International Business Machines Corporation. At that time we adopted Filon's numerical method <sup>(3)</sup>, based on parabolic interpolation and we obtained very accurate curves in a relatively short time.

Another way of getting practical results consists in replacing the  $G$  and the  $H$ -functions by suitably chosen approximations. The simplest method starts from the formal representation of  $\varrho$ ,  $i_z$  and  $i_x$  obtained by introducing the exponential terms corresponding to the first collision under the integral signs:

$$(92) \quad \varrho(r, \alpha) \rightarrow \frac{S_0 \lambda_s}{4\pi \bar{v} r} \int_0^\infty g_0^{(0,0)}(u) \sin ru \, du + \frac{S_1 \lambda_s \cos \alpha}{4\pi \bar{v} r} \int_0^\infty \frac{g_1^{(0,0)}(u)}{A} j_{\frac{3}{2}}(ru) \, du,$$

$$(93) \quad i_z(r, \alpha) \rightarrow \frac{S_0 \lambda_s}{4\pi r} \int_0^\infty g_0^{(1,0)}(u) j_{\frac{3}{2}}(ru) \, du + \\ + \frac{S_1 \lambda_s \cos \alpha}{4\pi r} \left[ \frac{2}{3A} \int_0^\infty g_1^{(1,0)}(u) j_{\frac{3}{2}}(ru) \, du - \frac{1}{3A} \int_0^\infty h_1^{(1,0)}(u) \sin ru \, du \right],$$

$$(94) \quad i_x(r, \alpha) \rightarrow \frac{S_1 \lambda_s \sin \alpha}{4\pi r} \cdot \frac{1}{3A} \left[ \int_0^\infty g_1^{(1,0)}(u) j_{\frac{3}{2}}(ru) \, du + \int_0^\infty h_1^{(1,0)}(u) \sin ru \, du \right],$$

in which

$$(95) \quad g_0^{(0,0)}(u) = \frac{2u}{\pi} \cdot \frac{\operatorname{Arctg} \lambda u - \frac{3b}{\lambda_s u} \left( 1 - \frac{\operatorname{Arctg} \lambda u}{\lambda u} \right)}{\lambda_s u - \operatorname{Arctg} \lambda u - \frac{3b}{\lambda_c u} \left( 1 - \frac{\operatorname{Arctg} \lambda u}{\lambda u} \right)},$$

$$(96) \quad g_0^{(1,0)}(u) = \frac{g_1^{(0,0)}(u)}{A} = \frac{2u}{\pi} \cdot \frac{\left( 1 - \frac{\operatorname{Arctg} \lambda u}{\lambda u} \right)}{\lambda_s u - \operatorname{Arctg} \lambda u - \frac{3b}{\lambda_c u} \left( 1 - \frac{\operatorname{Arctg} \lambda u}{\lambda u} \right)},$$

<sup>(3)</sup> Cfr. C. J. TRÄNTER: *Integral Transforms in Mathematical Physics*, London, p. 67

$$(97) \quad \frac{g_1^{(1,0)}(u)}{A} = \frac{2u}{\pi} \left\{ \frac{\text{Arctg } \lambda u + \frac{1}{\lambda u} \left( \frac{\text{Arctg } \lambda u}{\lambda u} - 1 \right)}{2\lambda_s u - 3b \left[ \text{Arctg } \lambda u + \frac{1}{\lambda u} \left( \frac{\text{Arctg } \lambda u}{\lambda u} - 1 \right) \right]} - \frac{\frac{1}{\lambda_c u} \left( 1 - \frac{\text{Arctg } \lambda u}{\lambda u} \right)}{\lambda_s u - \text{Arctg } \lambda u - \frac{3b}{\lambda_c u} \left( 1 - \frac{\text{Arctg } \lambda u}{\lambda u} \right)} \right\},$$

$$(98) \quad \frac{h_1^{(1,0)}(u)}{A} = -\frac{2u}{\pi} \left\{ \frac{2 \left[ \text{Arctg } \lambda u + \frac{1}{\lambda u} \left( \frac{\text{Arctg } \lambda u}{\lambda u} - 1 \right) \right]}{2\lambda_s u - 3b \left[ \text{Arctg } \lambda u + \frac{1}{\lambda u} \left( \frac{\text{Arctg } \lambda u}{\lambda u} - 1 \right) \right]} + \frac{\frac{1}{\lambda_c u} \left( 1 - \frac{\text{Arctg } \lambda u}{\lambda u} \right)}{\lambda_s u - \text{Arctg } \lambda u - \frac{3b}{\lambda_c u} \left( 1 - \frac{\text{Arctg } \lambda u}{\lambda u} \right)} \right\}.$$

Making use of the series expansion of  $\text{Arctg } \lambda u$ , it is possible to replace the preceding expressions by simple approximations, which are particularly valid in the region of small  $\lambda u$ . As an example, we can write for  $g_0^{(0,0)}(u)$ :

$$(99) \quad g_0^{(0,0)}(u) = \frac{2u}{\pi} \cdot \frac{\left( 1 - \frac{b\lambda}{\lambda_s} \right) - \lambda^2 u^2 \left( \frac{1}{3} - \frac{3b\lambda}{5\lambda_s} \right) + O(u^4)}{\frac{\lambda_s}{\lambda_c} \left( 1 - \frac{b\lambda}{\lambda_s} \right) + \lambda^2 u^2 \left( \frac{1}{3} + \frac{3b\lambda}{5\lambda_c} \right) + O(u^4)} =$$

$$= \frac{2u\lambda_c}{\pi\lambda_s} \cdot \frac{1}{\left[ 1 + \frac{\lambda^2 u^2}{\left( 1 - \frac{b\lambda}{\lambda_s} \right)} \left( \frac{\lambda_c}{3\lambda_s} + \frac{3b\lambda}{5\lambda_s} \right) + O(u^4) \right] \left[ 1 + \frac{\lambda^2 u^2}{\left( 1 - \frac{b\lambda}{\lambda_s} \right)} \left( \frac{1}{3} - \frac{3b\lambda}{5\lambda_s} \right) + O(u^4) \right]} =$$

$$= \frac{2u\lambda_c}{\pi\lambda_s} \cdot \frac{1}{\left[ 1 + \frac{\lambda_c \lambda u^2}{3 \left( 1 - \frac{b\lambda}{\lambda_s} \right)} + O(u^4) \right]} \cong \frac{2u\lambda_c}{\pi\lambda_s} \cdot \frac{1}{(1 + L^2 u^2)},$$

in which

$$L^2 \equiv \frac{\lambda\lambda_c}{3 \left( 1 - \frac{b\lambda}{\lambda_s} \right)}.$$



In a similar way, we can choose the following approximations:

$$(100) \quad g_0^{(1,0)}(u) = \frac{g_1^{(0,0)}(u)}{A} \cong \frac{2L^2 u^2}{\pi \lambda_s (1 + L^2 u^2)},$$

$$(101) \quad \frac{g_1^{(1,0)}(u)}{A} \cong \frac{2u}{\pi} \left[ \frac{5l^2}{3\lambda_s \lambda (1 + l^2 u^2)} - \frac{L^2}{\lambda_s \lambda_c (1 + L^2 u^2)} \right],$$

$$(102) \quad \frac{h_1^{(1,0)}(u)}{A} \cong -\frac{2u}{\pi} \left[ \frac{10l^2}{3\lambda_s \lambda (1 + l^2 u^2)} + \frac{L^2}{\lambda_s \lambda_c (1 + L^2 u^2)} \right],$$

in which

$$l^2 = \frac{\lambda^2}{5 \left( 1 - \frac{b\lambda}{\lambda_s} \right)}.$$

Putting these functions into (92), (93) and (94), we can integrate:

$$(103) \quad \int_0^\infty g_0^{(0,0)}(u) \sin ru \, du \cong \frac{2\lambda_c}{\pi \lambda_s L^2} \int_0^\infty \frac{u \sin ru}{((1/L^2) + u^2)} \, du = \frac{\lambda_c}{\lambda_s L^2} \exp \left[ -\frac{r}{L} \right],$$

$$(104) \quad \int_0^\infty g_0^{(1,0)}(u) j_{\frac{1}{2}}(ru) \, du \cong \frac{2}{\pi \lambda_s} \int_0^\infty \frac{u^2}{((1/L^2) + u^2)} \left( \frac{\sin ru}{ru} - \cos ru \right) \, du = \\ = \left( \frac{1}{r} + \frac{1}{L} \right) \frac{\exp[-r/L]}{\lambda_s} - \frac{2}{\lambda_s} \delta(r), \text{ etc.}$$

The approximate solutions are:

$$(105) \quad \varrho(r, \alpha) \cong \frac{S_0 \lambda_c}{4\pi \bar{v} L^2} \cdot \frac{\exp[-r/L]}{r} + \frac{S_1 \cos \alpha}{4\pi \bar{v} L} \cdot \frac{\exp[-r/L]}{r} \left( 1 + \frac{L}{r} \right),$$

$$(106) \quad i_z(r, \alpha) \cong \frac{S_0}{4\pi L} \cdot \frac{\exp[-r/L]}{r} \left( 1 + \frac{L}{r} \right) + \\ + \frac{S_1 \cos \alpha}{4\pi \lambda_c} \cdot \frac{\exp[-r/L]}{r} \left( 1 + \frac{2L}{r} + \frac{2L^2}{r^2} \right) - \\ - \frac{5S_1 l \cos \alpha}{6\pi \lambda} \cdot \frac{\exp[-r/l]}{r^2} \left( 1 + \frac{l}{r} \right),$$

$$(107) \quad i_x(r, \alpha) \cong \\ \cong \frac{S_1 L \sin \alpha}{4\pi \lambda_c} \cdot \frac{\exp[-r/L]}{r^2} \left( 1 + \frac{L}{r} \right) - \frac{5S_1 \sin \alpha}{12\pi \lambda} \cdot \frac{\exp[-r/l]}{r} \left( 1 + \frac{l}{r} + \frac{l^2}{r^2} \right) \quad (r > 0).$$

Comparing the approximations (99) to (102) with the exact curves for  $\lambda u \gg 1$ , we see that (100) and the second terms in (101) and (102) are matching the exact curves very well. This cannot be said of (99) and the first terms in (101) and (102) which are converging to zero in contradiction to the exact curves. The effect on the approximations (105) to (107) is that the first term in (105) and the last terms in (106) and (107) are deviating very much from the exact distributions in the region of  $r$  smaller than a few mean free paths  $\lambda$ . Indeed, let us consider e.g. (103). The approximation which we have substituted for  $g_0^{(0,0)}(u)$  is only accurate in the region  $\lambda u \ll 1$ . If  $r$  becomes of the order of  $\lambda$ , the parameter  $u$  becomes equal to  $\pi/2\lambda$  at the first maximum of the sine function and this makes  $\lambda u$  of the order  $\pi/2$  at that maximum. Therefore, the main contribution to the Fourier integral is in this case subjected to some inaccuracy. This is in fact the reason why the approximate density  $\varrho$  and the current component  $i_z$  do not completely satisfy the boundary conditions:

$$(108) \quad \begin{cases} \lim_{r \rightarrow 0} 2\pi r^2 \varrho(r, \alpha) \sin \alpha \, d\alpha = \frac{1}{v} (S_0 + S_1 \cos \alpha) \frac{\sin \alpha \, d\alpha}{2}, \\ \lim_{r \rightarrow 0} 2\pi r^2 i_z(r, \alpha) \sin \alpha \, d\alpha = (S_0 + S_1 \cos \alpha) \frac{\sin \alpha \, d\alpha}{2} \end{cases}$$

as the exact curves (85) and (86) do. Therefore the results (105) to (107) are only applicable at distances larger than a few mean free paths  $\lambda$ . This implies that we are considering the cases for which capture is not too strong, because otherwise the density and the current would become very small at distances larger than a few mean free paths and the most interesting region to be studied would then be situated in the vicinity of the source.

Hence our conclusion is that with the applied method, we get results which are applicable when  $\lambda_c \gg \lambda_s$  and for  $r$  larger than a few mean free paths. At such distances the terms depending on  $l$  become quite unimportant with respect to the other terms. Considering the preceding conditions and the fact that in this problem the single scattering is not too forward, it is clear that the obtained approximations must be in agreement with diffusion theory. This is indeed the case for the terms depending on  $L$ . The density (105) is the only acceptable solution of the equation

$$(109) \quad D \nabla^2 \varrho - \frac{\bar{v} \varrho}{\lambda_c} = 0, \quad (r > 0)$$

in which we have

$$(110) \quad D = \frac{\lambda \bar{v}}{3 \left(1 - \frac{b\lambda}{\lambda_s}\right)} = \frac{\lambda \bar{v}}{3 \left(1 - \frac{\lambda}{\lambda_s} \cos \gamma\right)}.$$

In addition to this, we see that

$$(111) \quad \mathbf{i} = -D \operatorname{grad} \varrho$$

is satisfied. The diffusion constant and the diffusion length can be written as

$$(112) \quad D = \frac{\lambda_{\text{tr}} \bar{v}}{3}; \quad L = \left( \frac{\lambda_{\text{tr}} \lambda_c}{3} \right)^{1/2},$$

where the transport mean free path  $\lambda_{\text{tr}}$  is given by

$$(113) \quad \lambda_{\text{tr}} = \frac{\lambda}{\left(1 - \frac{\lambda}{\lambda_s} \cos \gamma\right)} = \frac{\lambda_s}{\left(1 + \frac{\lambda_s}{\lambda_c} \cos \gamma\right)}.$$

This is slightly different from the usually accepted formula, which is

$$(114) \quad \lambda_{\text{tr}} = \frac{\lambda_s}{(1 - \cos \gamma)},$$

but in practice, the difference between (114) and (113) is quite unimportant because  $\lambda_s/\lambda_c$  is small compared to unity. Moreover, (113) is a result which is indirectly deduced from a simple approximation method leading to the formulae (105) to (107), and therefore, it must not be regarded as the best representation of  $\lambda_{\text{tr}}$ . This will be discussed further on. What we have really tried to show is the connection between the results obtained from the exact theory and those satisfying the diffusion theory.

Taking (113) as the expression for  $\lambda_{\text{tr}}$ , it turns out that the first part of (105) is exact up to the second order moment. Indeed, if we call that first part  $\varrho^{(0)}(r)$  and if we calculate

$$(115) \quad \iiint_{-\infty}^{\infty} \varrho^{(0)}(r) dV \quad \text{and} \quad \iiint_{-\infty}^{\infty} r^2 \varrho^{(0)}(r) dV,$$

we obtain results which are identical to those calculated by means of the exact curves. This is in fact a logical consequence of our approximation procedure (99). Inversely, we can say that if we take the solution of the diffusion equation

$$(116) \quad \varrho^{(0)}(r) = C \frac{\exp[-r/L]}{r}$$

and we calculate the coefficients  $C$  and  $L$  from the identification of the integrals (115) with their exact values, we get identically the first part of (105) with  $\lambda_{tr}$  given by (113). However, due to the fact that  $\varrho^{(0)}(r)$  is not strictly valid in the region of  $r$  smaller than a few times  $\lambda$ , we see that the identification procedure is not completely justified. Strictly speaking, the identification of the integrals can only be done when the region of a few times  $\lambda$  around the source is excluded from the integration space in (115). This would probably introduce a slight change into the expression for  $\lambda_{tr}$ , but it would be a difficult thing to calculate.

It is much more interesting to try to modify the approximations (105) to (107) so as to make them valid for small  $r$ . We have seen that the difficulty comes from the fact that (99) and the first terms in (101) and (102) are deviating from the exact curves when  $u$  goes to infinity. This can be eliminated by approximating the more complicated functions  $G_0^{(0)}(u)$ ,  $G_1^{(0)}(u)$  and  $H_1^{(0)}(u)$  instead of the corresponding  $g$  and  $h$ -functions, and by using the first part of (85), the last part of (86) and the entire formula (87) instead of the corresponding terms in (92), (93) and (94). In this way, we find for instance for the density:

$$(117) \quad \varrho(r, \alpha) \cong \frac{S_0}{4\pi\bar{v}} \cdot \frac{\exp[-r/\lambda]}{r^2} + \\ + \frac{S_0\lambda_c\lambda}{4\pi\bar{v}\lambda_s L'^2} \cdot \frac{\exp[-r/L']}{r} + \frac{S_1 \cos \alpha}{4\pi\bar{v}L} \cdot \frac{\exp[-r/L]}{r} \left(1 + \frac{L}{r}\right),$$

in which

$$L^2 \equiv \frac{\lambda\lambda_c}{3\left(1 - \frac{b\lambda}{\lambda_s}\right)}; \quad L'^2 \equiv \frac{\lambda\lambda_c}{3\left(1 - \frac{b\lambda}{\lambda_s}\right)} \left(1 + \frac{\lambda}{\lambda_c} + b \frac{\lambda^2}{\lambda_c^2}\right).$$

This is the approximation corrected for small distances and satisfying the first boundary condition (108). The expression for  $L'$  corresponds more closely to a transport mean free path given by (114) than  $L$ .

We wish to express our sincere thanks to the Institut Interuniversitaire des Sciences Nucléaires and to Prof. Dr. J. L. VERHAEGHE for his interest in this work.



## RIASSUNTO (\*)

In relazione al calcolo della perturbazione prodotta in una corrente di neutroni termici da un sottile rivelatore circolare, abbiamo risolto il seguente problema. È data una sorgente puntiforme la cui intensità è descritta da una distribuzione isotropica più un termine cosinusoidale non isotropico. La legge che governa gli scattering singoli contiene anch'essa un termine non isotropico. Dobbiamo calcolare la densità neutronica e la distribuzione delle direzioni della velocità in ogni punto dello spazio. Mantenendo, per ragioni di generalità, arbitrarie le probabilità dei singoli percorsi, abbiamo stabilito delle esatte relazioni ricorrenti tra le probabilità di distribuzione precedentemente definite. Sviluppando queste funzioni in serie di armoniche sferiche e impiegando, inoltre, alcuni teoremi di addizione per le funzioni di Bessel e i polinomi di Legendre è stato possibile di trasformare le relazioni integrali in semplici equazioni lineari. I risultati finali si ottengono sotto forma di integrali di cui si possono ottenere i valori numerici coll'impiego di macchine calcolatrici. Introducendo funzioni semplificate sotto i segni d'integrazione è anche possibile ottenere buone approssimazioni per la densità e la corrente neutronica. Si discute dettagliatamente un semplice esempio di questo metodo e si confrontano i risultati con quelli offerti dalla teoria della diffusione.

(\*) Traduzione a cura della Redazione.

## Sullo spettro della componente fotonica al livello del mare.

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(ricevuto il 5 Novembre 1953)

**Riassunto.** — Un dispositivo ad anticoincidenza viene usato per misurare l'assorbimento dei prodotti di materializzazione dei fotoni della componente fotonica al livello del mare. Si istituisce un calcolo, usando la teoria degli sciami secondo l'approssimazione « B » di Rossi, per confrontare i dati sperimentali con la teoria. Si trova che il miglior adattamento si ottiene impiegando per la fotonica uno spettro differenziale di potenza con esponente 2,1.

### 1. — Introduzione.

È solo da qualche anno che si conosce con sufficiente approssimazione la genesi della componente elettrofotonica nell'atmosfera. Infatti, mentre era ben noto che una parte, preponderante al livello del mare, della componente elettrofotonica nasce dai mesoni  $\mu$ , sia per disintegrazione sia per collisione, permanevano seri dubbi per la parte rimanente che diventa invece importante nell'alta atmosfera. Questi dubbi vennero chiariti quando si constatò l'esistenza dei mesoni neutri che si disintegrano in due fotoni; da questo processo può infatti trarre origine quella parte dell'elettrofotonica che eccede il contributo dato dai mesoni  $\mu$ .

I dati sperimentali su questa componente sono abbondanti, ma quando si considerano separatamente le componenti pure, essi diventano scarsi, in special modo nei riguardi della fotonica, e ciò a seguito dell'intrinseca difficoltà della rivelazione di detta componente, che avviene sempre in modo indiretto.

Ci è sembrato pertanto interessante istituire una serie di esperienze aventi lo scopo di aumentare la conoscenza dei dati sperimentali sulla fotonica nella bassa atmosfera, e per controllare se essi si adattino agli schemi preceden-

temente esposti. E poichè uno dei punti dubbi che permangono sulla fotonica al livello del mare è quello riguardante il suo spettro, vogliamo iniziare la nostra indagine con un'esperienza che valga a dare qualche indicazione in proposito.

È noto che lo spettro della fotonica è stato sempre considerato del tipo di potenza, esprimibile, secondo quanto originariamente suggerito da HEITLER e NORDHEIM <sup>(1)</sup>, con:

$$\left. \begin{aligned} \varrho(W) dW &= K \frac{dW}{W} && \text{per } W < W_c \\ &= K \left( \frac{W_c}{W} \right)^\alpha \frac{dW}{W} && \text{per } W > W_c \end{aligned} \right\} \quad \text{con } W_c = 150 \text{ MeV in aria}$$

JÁNOSSY e ROSSI <sup>(2)</sup> hanno per primi cercato di valutare sperimentalmente l'esponente  $\alpha$ : paragonando le curve sperimentali degli sciami sotto piombo con quelle teoriche essi hanno trovato il miglior accordo assegnando ad  $\alpha$  il valore di 1,5. Successivamente J. CLAY <sup>(3)</sup> e C. N. CHOU <sup>(4)</sup> hanno ripreso il problema: quest'ultimo, con un dispositivo misto a contatori ed a scintillatori plastici, ha trovato per  $\alpha$  un valore pari ad 1,7 per le alte energie, mentre sembra che per le energie più basse esso diminuisca.

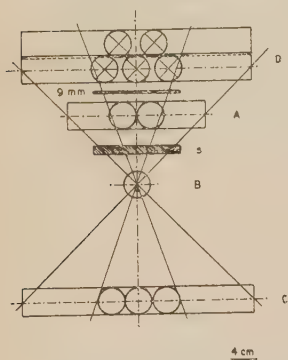


Fig. 1. — Dispositivo sperimentale. I contatori dei piatti A, B, C sono in coincidenza tripla; quelli del piatto D sono in anticoincidenza con i precedenti.

## 2. — Dispositivo sperimentale.

Il dispositivo sperimentale assomiglia a quello precedentemente usato da JÁNOSSY e ROSSI <sup>(2)</sup> e consiste in un telescopio di contatori a coincidenze triple A, B, C, su cui è disposto un piatto D di contatori in anticoincidenza (fig. 1). I fotoni in arrivo si materializzano nello strato di piombo sottostante il piatto superiore D ed i prodotti di materializzazione vengono assorbiti da uno strato di piombo a spessore variabile  $s$ , inserito fra i contatori del telescopio. In questo modo l'assorbimento invece di venir effettuato direttamente sui fotoni, viene

(1) W. HEITLER: *Proc. Roy. Soc.*, A **161**, 261 (1937); L. W. NORDHEIM e M. H. WEBB: *Phys. Rev.*, **56**, 494 (1939).

(2) L. JÁNOSSY e B. ROSSI: *Proc. Phys. Soc.*, A, **175** (1940).

(3) J. CLAY: *Physica*, **14**, 569 (1949).

(4) C. N. CHOU: *Phys. Rev.*, **90**, 473 (1953).

effettuato sui prodotti di materializzazione. Un circuito elettronico a buon potere risolutivo ( $2 \mu s$ ) consente la registrazione simultanea delle coincidenze triple e delle anticoincidenze. Queste ultime sono quindi rappresentate da un evento costituito dal passaggio di un fotone che si materializza nello sciamatore ed i cui prodotti di materializzazione hanno energia sufficiente a superare lo strato dell'assorbitore  $s_1$  e venir rivelati.

La valutazione del fondo è stata ottenuta disponendo al di sopra del piatto di anticoincidenza uno strato di piombo di 8 cm. Poichè la probabilità che un fotone attraversi un tale assorbitore senza materializzarsi è molto piccola, il numero di eventi che il dispositivo registra in tali condizioni si può ritenere senz'altro come dovuto al fondo. Si è trovato per esso un valore di 12 impulsi all'ora, e questo valore è stato sottratto a tutte le misure ottenute in fase di esperienza. I dati sperimentali così corretti sono stati riportati, in funzione dello spessore dell'assorbitore, nel diagramma semilogaritmico della fig. 2, e sono stati segnati con punti iscritti in cerchietti. L'errore statistico, per ciascuna misura, non eccede il diametro dei cerchietti.

### 3. - Analisi teorica dell'esperienza.

Vogliamo ora procedere ad un paragone fra i dati sperimentali così ottenuti e quelli che si ricavano da un'analisi teorica del processo. Procediamo in due tempi successivi: valutiamo anzitutto il numero di elettroni che fuoriescono dallo sciamatore e sono contenuti nell'angolo solido del telescopio e calcoliamo successivamente l'assorbimento che essi subiscono quando attraversano lo strato assorbente di piombo. In tutto il ragionamento che segue si farà uso della teoria unidimensionale degli sciame, secondo l'approssimazione « B » di ROSSI <sup>(5)</sup>, e pertanto le notazioni e le formule usate sono quelle del citato lavoro.

Assumiamo, seguendo ROSSI <sup>(2)</sup>, che lo sciame generato da un fotone nello

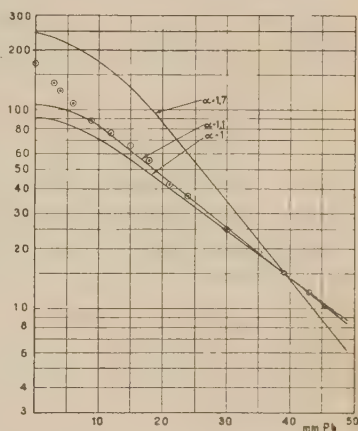


Fig. 2. - Assorbimento dei prodotti di materializzazione dei fotoni nel piombo. Ascisse: spessori dell'assorbitore di piombo in mm; ordinate: numero di anticoincidenze all'ora;  $\odot$ : punti sperimentali; le curve teoriche sono a tratto continuo e sono state calcolate variando l'esponente  $\alpha+1$  dello spettro fotonico.

<sup>(5)</sup> B. ROSSI e K. GREISEN: *Rev. Mod. Phys.*, **13**, 240 (1941).



sciamatore abbia un'apertura media  $\Omega$ , nella quale gli elettroni siano distribuiti a caso; sia invece  $\omega$  l'angolo solido determinato dai contatori inferiori del telescopio, visti da un punto dei contatori inferiori e sia  $\omega \ll \Omega$ . Supponiamo anzitutto che non sia inserito fra i contatori l'assorbitore  $x$ . Sia  $\Pi(W, E_0, x_0)$  il numero totale medio di elettroni generati da un primario fotografico di energia  $W$  nello sciamatore  $x_0$ , e che abbiano almeno l'energia minima necessaria per superare le pareti dei contatori. La probabilità che un fotone produca un'anticoincidenza è data allora da:

$$p = 1 - \exp [\omega/\Omega] \Pi(W, E_0, x_0),$$

che per  $(\omega/\Omega)\Pi \ll 1$  <sup>(2)</sup> è dato semplicemente da:  $(\omega/\Omega)\Pi$ .

Se  $F$  è la superficie effettiva di generazione degli sciami ed  $J$  l'intensità fotonica direzionale, allora il numero di sciami rivelati per unità di tempo è:

$$J_s = J \cdot \Omega \cdot F \cdot p.$$

Invece il numero di fotoni che attraversa l'angolo solido del telescopio è:

$$J_p = J \cdot \omega \cdot F,$$

È utile ora considerare il rapporto:

$$(3) \quad P(W, x_0) = J_s/J_p = \frac{J \cdot \Omega \cdot F \cdot p}{J \cdot \omega \cdot F} = \frac{\Omega}{\omega} p = \frac{\Omega}{\omega} \left( 1 - \exp \left[ - \frac{\omega}{\Omega} \right] \Pi \right) \\ \simeq \Pi(W, E_0, x_0) \quad \text{per } (\omega/\Omega)\Pi \ll 1.$$

Per piccoli sciami la limitazione (2) è senz'altro verificata; per sciami più grandi possiamo prendere, con grossolana approssimazione,  $\Omega = 2\pi$ , allora, tenendo conto della geometria del nostro dispositivo e dei dati di Arley <sup>(6)</sup>, si può ritenere che la (2) resti valida fino ad energie dell'ordine di  $10^9$  eV; questo valore costituisce pertanto un limite superiore alla validità del nostro calcolo.

Notiamo però che fotoni molto energetici sono molto rari e pertanto la limitazione introdotta non può modificare apprezzabilmente i risultati, specialmente per la parte di bassa e media energia dello spettro.

Sia ora  $\varrho(W)dW$  lo spettro fotonico differenziale dei fotoni primari, almeno di un fattore di normalizzazione, il numero di eventi rivelati dal dispositivo, è dato da:

$$T(x_0) = \int_{W_0}^{\infty} \varrho(W) P(W, x_0) dW = \int_{E_0}^{\infty} \varrho(W) \Pi(W, E_0, x_0) dW,$$

(6) D. I. X. MONTGOMERY: *Cosmic Rays* (Princeton, 1949), p. 335.

dove il limite inferiore  $W_0 \simeq E_0$  indica il minimo valore dell'energia che debbono avere gli elettroni di materializzazione e quindi i primari fotonici per poter attraversare le pareti dei contatori e venir quindi rivelati.

Consideriamo ora l'effetto dell'assorbitore  $x$ , inserito nel telescopio. Dal punto di vista del computo degli eventi rivelati, esso implica la seguente sostituzione:

$$\Pi(W, E_0, x_0) = \int_{E_0}^{\infty} \pi(W, E, x_0) \omega(E, x) dE$$

dove  $\omega(E, x)$  rappresenta la probabilità che ha un elettrone di energia  $E$  di superare l'assorbitore di piombo di spessore  $x$ . Il calcolo di questa probabilità non è noto; si può con approssimazione ritenere che essa sia uguale ad 1 per tutti gli elettroni che, uscendo da  $x_0$ , posseggano un'energia  $E(x)$  sufficiente a creare sciami di almeno un elettrone all'uscita dell'assorbitore  $x$ , mentre sia uguale a 0, se l'energia di detti elettroni è minore di  $E(x)$ . L'energia  $E(x)$  in tal caso viene ad assumere il significato di un'energia necessaria agli elettroni per possedere il percorso (range)  $x$  nel piombo. I valori di  $E(x)$  si possono ricavare estrapolando ad  $n=1$  le curve che danno l'energia del primario elettronico richiesta per produrre sciami con un numero totale  $n$  di elettroni, in funzione dello spessore di materiale attraversato, calcolate secondo il metodo di Snyder e Serber (7).

Allora il numero di eventi registrati dal dispositivo in funzione di  $x$ , sarà dato da:

$$T(x_0, x) = \int_{E_0}^{\infty} dW \int_{E(x)}^{\infty} \varrho(W) \pi(W, E(x), x_0) \omega(E, x) dE.$$

È evidente ora, che fissato lo spessore dell'assorbitore  $x$ , resta determinata l'energia minima  $E_1(x)$  che deve possedere l'elettrone per attraversarlo; se l'energia dell'elettrone risulta  $E > E_1(x)$ , sarà  $\omega(E, x) = 1$  altrimenti, per  $E < E_1(x)$ , sarà  $\omega(E, x) = 0$ . Ciò equivale a scrivere la precedente relazione nel seguente modo:

$$T(x_0, x) = \int_{E_0}^{\infty} dW \int_{E_1(x)}^{\infty} dE \varrho(W) \pi(W, E, x_0).$$

Dalle curve energia percorso ora menzionate, invece della probabilità si

(7) B. ROSSI: *High Energy Particles*. New York, 1952, p. 259.

può ricavare l'energia minima che è necessaria all'elettrone per attraversare l'assorbitore di determinato spessore  $x$ , e questa energia va posta al limite inferiore dell'integrale. A questo proposito c'è da aggiungere ancora che se il fotone primario ha un'energia  $W$  tale che  $W_0 < W < E_1(x)$ , esso genera elettroni all'uscita di  $x_0$ , ma questi non sono più capaci di attraversare lo strato successivo  $x$ . Al limite ciò può accadere solo se  $W = E_1(x)$ . Quindi anche il primo limite inferiore può essere scritto  $E_1(x)$  ed in definitiva si ottiene:

$$(3) \quad T(x_0, x) = \int_{E_1(x)}^{\infty} dW \int_{E_1(x)}^{\infty} dE \varrho(W) \pi(W, E, x_0).$$

Per il calcolo di questa funzione conviene ricorrere alla sua trasformata di Laplace:

$$(4) \quad \mathcal{L}_T(\lambda_0) = \int_{E_1(x)}^{\infty} dW \int_{E_1(x)}^{\infty} dE \varrho(W) L_{\pi}(W, E, \lambda_0),$$

e prendere per  $L_{\pi}$  il valore calcolato secondo l'approssimazione « B » di Rossi, con  $2\varepsilon_0 < E < W$  (\*). La limitazione inferiore è soddisfatta, non quella superiore; è da osservare tuttavia che quando  $E \lesssim W$ ,  $L_{\pi}$  è molto piccola, mentre assume valori di gran lunga maggiori quando  $E \ll W$ , quando cioè l'approssimazione « B » diventa valida. Perciò con una certa approssimazione prendiamo:

$$L_{\pi}(W, E, \lambda_0) = - \frac{\varphi(s)}{\lambda_1'(s)} \frac{W^s}{\{\mu_1(s, \varepsilon_0/E)\}^{s+1}} \quad \text{con} \quad \varphi(s) = - \frac{B(s)}{\lambda_1(s) - \lambda_2(s)}.$$

Sostituiamo ora nella (4) il valore di  $L_{\pi}(W, E, \lambda_0)$ , l'espressione (1) dello spettro differenziale fotonico, ed integriamo successivamente su  $W$  e su  $E$ , ottenendo:

$$\begin{aligned} \mathcal{L}_T &= K \frac{\varphi(s)}{\lambda_1'(s)} \frac{\alpha}{s^2(\alpha - s)} \frac{1}{\{\mu_1(s, \varepsilon_0/E)\}^{s+1}} \left(\frac{W_c}{E_1}\right)^s \quad \text{per } E_1 \leq W_c \\ &= K \frac{\varphi(s)}{\lambda_1'(s)} \frac{1}{s(\alpha - s)} \frac{1}{\{\mu_1(s, \varepsilon_0/E)\}^{s+1}} \left(\frac{W_c}{E_1}\right)^{\alpha} \quad \text{per } E_1 > W_c. \end{aligned}$$

Per il calcolo di questi integrali ci siamo valse delle seguenti due semplificazioni. Nell'espressione  $(W_c/E_1)^s - 1$ , che compare nella (4) quando  $E_1 \leq W_c$ , abbiamo ammesso che sia  $W_c \gg E_1$ , di modo che si possa assumere  $(W_c/E_1)^s - 1 \cong (W_c/E_1)^s$ . In entrambi i casi, sia per  $E_1 < W_c$ , che per  $E_1 > W_c$ , abbiamo ritenuto la funzione  $\mu_1(s, \varepsilon_0/E)$  variabile molto lentamente con  $E$ .

(\*)  $\varepsilon_0$  è l'energia critica per il Pb, che abbiamo preso uguale a 7,6 MeV (vedi (7) p. 295).

Le due trasformate si possono calcolare facilmente col metodo della sella e si ottiene:

$$\left. \begin{aligned} T_1(x_0, x) &= \frac{K}{\sqrt{2\pi}} \frac{\alpha \varphi(s)}{s^2(\alpha-s)} \frac{1}{\{D''(s)\}^{1/2}} \left( \frac{W_c}{E_1} \right)^s \frac{\exp[\lambda_1(s)x_0]}{\{\mu_1(s, \varepsilon_0/E)\}^{s+1}} \\ D''(s) &= \lambda_1''(s)x_0 - \frac{4\alpha s - 3s^2 - 2\alpha^2}{s^2(\alpha-s)^2}; \quad \text{sella: } \lambda_1'(s)x_0 - \ln \frac{W_c}{E_1} - \frac{2\alpha - 3s}{(\alpha-s)} = 0 \end{aligned} \right\} E_1 \leq W_c$$

$$\left. \begin{aligned} T_2(x_0, x) &= \frac{K}{\sqrt{2\pi}} \frac{1}{s(\alpha-s)} \frac{\varphi(s)}{\{D''(s)\}^{1/2}} \left( \frac{W_c}{E_1} \right)^\alpha \frac{\exp[\lambda_1(s)x_0]}{\{\mu_1(s, \varepsilon_0/E_1)\}^{s+1}} \\ D''(s) &= \lambda_1''(s)x_0 - \frac{2\alpha s - \alpha^2 - s^2}{s^2(\alpha-s)^2}; \quad \text{sella: } \lambda_1'(s)x_0 - \frac{\alpha - 2s}{s(\alpha-s)} = 0. \end{aligned} \right\} E_1 > W_c$$

Naturalmente per  $E_1 = W_c$ , le due funzioni si eguagliano, assieme alle loro derivate, il che assicura la continuità delle curve relative, in corrispondenza al valore  $W_c$  dell'energia.

Si noti ancora che per  $E_1 > W_c$  la seconda funzione tenderebbe a comportarsi come la funzione  $1/E^\alpha$ ; salvo la lieve dipendenza da  $s$ , tramite la sella.

Concludendo, il numero teorico di eventi che si debbono registrare nel dispositivo, in funzione dello spessore, è dato da:

$$T(x_0, x) = T_1(x_0, x) + T_2(x_0, x).$$

#### 4. - Discussione e conclusione.

I risultati dei calcoli sono stati riportati nel diagramma della fig. 2, a mezzo delle curve continue, assegnando all'esponente  $\alpha$  tre valori distinti: 1; 1,1; 1,7. Il valore minimo dell'energia,  $E_1(x)$ , che si ha per  $x = 0$ , è stato preso uguale a 10 MeV, tale essendo l'energia che si suppone necessaria agli elettroni per attraversare soltanto i contatori del telescopio, con i relativi sostegni. Le curve sono normalizzate fra loro e con i dati sperimentali, in corrispondenza all'energia  $W_c = 150$  MeV, cui compete uno spessore di 39 mm di Pb per l'assorbitore. Il punto di normalizzazione è indicato in diagramma con un doppio cerchio.

Si può osservare come il miglior adattamento ai dati sperimentali si ottenga con la curva che si ricava impiegando per l'esponente dello spettro differenziale della fotonica il valore 2,1 ( $\alpha = 1,1$ ). Essa concorda perfettamente con i dati sperimentali a partire dallo spessore di 10 mm di Pb; la parte superiore non si adatta invece altrettanto bene. Questo comportamento è più che comprensibile quando si pensi che la teoria adoperata tiene conto in modo insufficiente, o addirittura non tien conto affatto, dei processi elementari alle basse energie, come la ionizzazione o l'effetto Compton. Comunque agli effetti della nostra ricerca è importante rilevare come la parte esponenziale dell'assor-



bimento, che è rappresentata nel diagramma con un andamento rettilineo, si adatti proprio a quella parte della curva teorica che è meno sensibile alle ipotesi e semplificazioni introdotte.

Il valore scelto per  $\alpha$  influisce sensibilmente sull'inclinazione della curva teorica: i due valori estremi di  $\alpha = 1$  ed  $\alpha = 1,7$  portano a tracciamenti notevolmente discosti; del resto anche la curva ottenuta con  $\alpha = 1,2$  (non tracciata in figura) risulta già chiaramente al disopra dei valori sperimentali. Questo permette di asserire che il valore trovato per  $\alpha$  è abbastanza ben definito. Esso tuttavia risulta alquanto inferiore ai valori precedentemente conosciuti. Ma a questo riguardo si possono fare le seguenti considerazioni.

Anzitutto, per quanto detto all'inizio, il valore di 2,1 per l'esponente dello spettro fotonico non è in contrasto con le attuali vedute sulla genesi della elettrofotonica, anzi esso viene a denunciare un più stretto legame con la componente nucleonica, come appunto era da attendersi. In secondo luogo la pendenza dello spettro dipende quasi sicuramente dalla banda energetica considerata, e quindi dal tipo di dispositivo sperimentale usato. Così C. N. CHOU, col suo dispositivo che consente di ricavare dati fino ad energie dell'ordine di 2 GeV, ottiene, basandosi piuttosto sulle alte energie, uno spettro con esponente 2,7. Ma se nel diagramma di CHOU ci si limita a considerare i dati corrispondenti alla banda energetica da 80 ad 800 MeV, si trova precisamente uno spettro con esponente 2,1. Il nostro dispositivo è viceversa adatto alla rivelazione di eventi di bassa e media energia, e quindi la risposta che con esso otteniamo è piuttosto da riferire a quest'ultima banda energetica. È da arguire quindi che, con grande probabilità, la pendenza dello spettro della fotonica non sia unica, ma vada crescendo al crescere dell'energia.

Desideriamo ringraziare sentitamente il Direttore dell'Istituto, Prof. A. GIACOMINI, per l'incoraggiamento datoci alla presente ricerca, ed il Dott. P. BUDINI per le interessanti discussioni avute sull'argomento.

## SUMMARY

The shape of the photonic spectrum of cosmic rays at sea level is obtained measuring the absorption of the photon materialization products in an anticoincidence device (fig. 1) and putting these data in comparison with those observed using the shower theory under Rossi's «B» approximation. A good agreement is found if one takes a power spectrum with exponent 2,1. As the experimental device is particularly sensitive to low and medium energies, one can infer that this value is in agreement with modern views of the electrophotonic component in the air, but in the meanwhile it is suggested that the spectrum exponent could not be the same in the whole energetic range, but rather variable as a function of the energy.

## Discriminatore differenziale oscillografico e sua applicazione nei circuiti a coincidenza.

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(ricevuto il 9 Novembre 1953)

**Riassunto.** — Si descrive un discriminatore differenziale in cui gli impulsi vengono presentati sullo schermo di un tubo oscillografico a raggi catodici, registrati fotograficamente ed esaminati al microfotometro. Il discriminatore viene utilizzato in un dispositivo a coincidenze.

In nostri lavori precedenti <sup>(1)</sup> è apparsa la necessità di distinguere tra le varie radiazioni  $\gamma$  emesse da certi nuclidi radioattivi in particolare dal  $^{214}_{84}\text{Po}$ .

Tali ricerche, riguardando le cascate di raggi  $\gamma$  (coincidenze), vennero eseguite in un primo tempo mediante misure di assorbimento, mentre in un secondo tempo venne costruito e utilizzato un circuito a coincidenze discriminate atto a contare soltanto le coincidenze corrispondenti a un certo valore dell'impulso di tensione generato da uno dei due fotomoltiplicatori, per effetto della scintillazione prodotta da un raggio  $\gamma$  di data energia.

Ma il tempo necessario per ottenere uno spettro complesso di un nuclide radioattivo quale ad esempio il  $^{214}_{84}\text{Po}$  sarebbe stato così lungo e sarebbe stato così difficile ottenere una sufficiente fedeltà da indurci a seguire altra via basata sul seguente concetto.

I raggi  $\gamma$  ricevuti dal cristallo possono generare delle scintillazioni che, raccolte dal fotomoltiplicatore, vengono trasformate in impulsi di tensione i quali, convenientemente amplificati, sono applicati al pennello elettronico di un oscillografo a raggi catodici.

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<sup>(1)</sup> F. DEMICHELIS e R. MALVANO: *Nuovo Cimento*, **9**, 1106 (1952); **10**, 405 (1953);  
F. DEMICHELIS e R. MALVANO: *Nuovo Cimento* **10**, 1359 (1953).

Come è noto, la distribuzione dell'altezza dei vari impulsi dovuti ai raggi  $\gamma$  di una determinata energia varia da zero fino a un valore massimo, questo ultimo essendo proporzionale all'energia della radiazione  $\gamma$  incidente.

Ma l'altezza dei vari impulsi si addensa intorno a certi valori legati ai diversi processi di interazione dei raggi  $\gamma$  con la materia, e sostanzialmente all'energia dei raggi  $\gamma$  stessi.

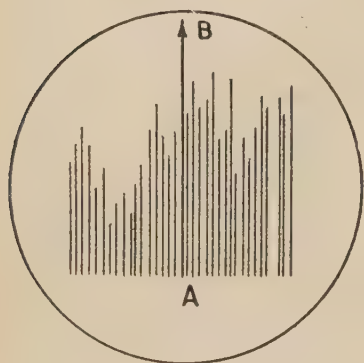


Fig. 1.

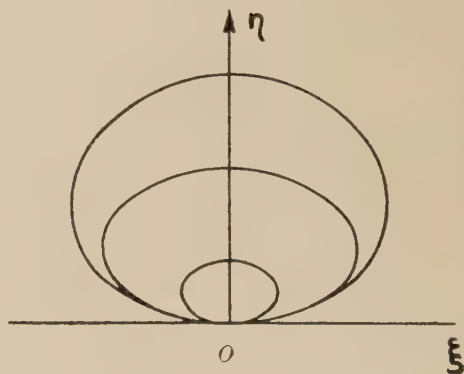


Fig. 2.

Portati semplicemente gli impulsi sull'asse verticale dell'oscillografo, sullo schermo appare una distribuzione del tipo di fig. 1, che, se fotografata con una posa opportuna, permette di eseguire un esame globale microfotometrico dell'annerimento mediante una conveniente fenditura esploratrice che si muove lungo l'asse verticale  $AB$  della fotografia stessa. Si verrebbe così ad ottenere uno spettro integrale degli impulsi <sup>(2)</sup>.

Per ottenere i vantaggi ben noti della discriminazione differenziale abbiamo proceduto nel modo seguente.

Si fa in modo che l'impulso proveniente dal fotomoltiplicatore venga traddotto sullo schermo dell'oscillografo in una curva chiusa come in fig. 2; tale curva appare più o meno grande proporzionalmente all'altezza dell'impulso stesso sempre partendo da  $O$  e restando tangente alla stessa retta  $\xi$ .

Per ottenere questo risultato l'impulso viene mandato a comandare la deviazione verticale del pennello elettronico mentre la deviazione orizzontale di esso viene comandata dalla derivata (nel tempo) dell'azione costituente l'impulso stesso.

<sup>(2)</sup> R. HOFSTÄDTER e J. A. MCINTYRE: *Nucleonics*, **7**, 3, 32 (1950); S. A. E. JOHANSSON e S. ALMQUIST: *Arkiv för Fysik*, **5**, 20, 427 (1952).

Il concetto non è nuovo <sup>(3)</sup>, ma non ci risulta che esso sia stato utilizzato unitamente alla registrazione fotografica seguita da una esplorazione microfotometrica di un tratto come  $O\eta$  di fig. 2.

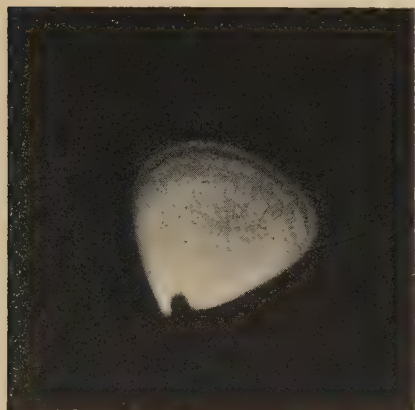


Fig. 3a.

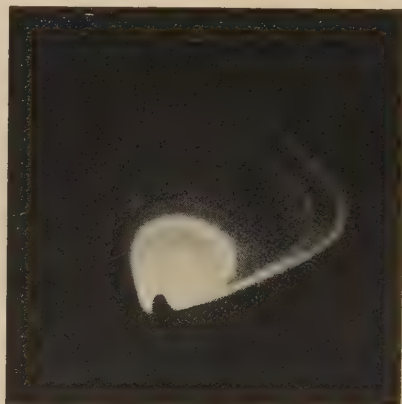


Fig. 3b.

I risultati ottenuti si sono rivelati estremamente promettenti: lungo  $O\eta$  si ha uno *spettrogramma* di raggi  $\gamma$  ed il dispositivo può considerarsi un discriminatore differenziale ad innumerevoli canali.

In fig. 3 sono riportate le fotografie relative agli spettrogrammi del  $^{60}_{27}\text{Co}$  e del  $^{228}_{90}\text{RdTh}$  in equilibrio con i suoi prodotti di decadimento.

In fig. 4 è la microfotometria relativa al  $^{60}_{27}\text{Co}$ .

Particolare applicazione del dispositivo è stata fatta utilizzando come spettrografo a coincidenze e cioè facendo in modo che, ogniqualevolta il circuito a coincidenze segnala l'arrivo di una coppia di raggi  $\gamma$ , un impulso di tensione opportuno inviato a comandare la griglia di intensità del tubo oscillografico rende

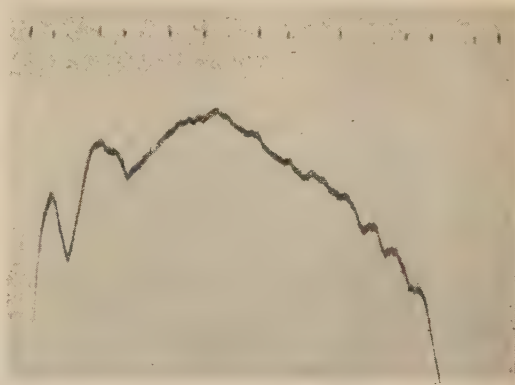


Fig. 4— Registrazione eseguita al microfotometro Zeiss. Apertura della fenditura:  $0,3 \times 0,08 \text{ mm}^2$ .

<sup>(3)</sup> G. T. WRIGHT: *Journ. Sc. Inst.*, **29**, 157 (1952).



visibile sullo schermo l'impulso corrispondente ad uno dei due raggi  $\gamma$  in coincidenza.

Nella fig. 5 è riportata la fotografia relativa allo spettro dei raggi  $\gamma$  in coincidenza provenienti dal decadimento del  $^{228}_{90}\text{RdTh}$  in equilibrio con i suoi sottoprodotti.

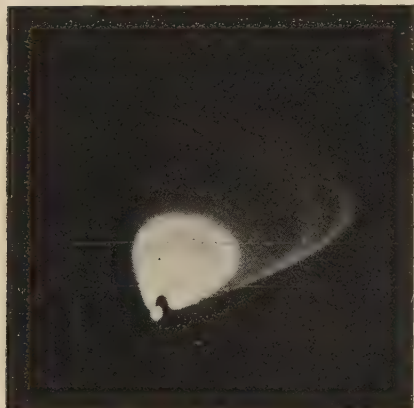


Fig. 5.

L'impiego del metodo da noi segnalato si presenta come atto a feconde utilizzazioni, sicchè, mentre si pone il problema di tradurlo su scala quantitativa, si è ritenuto già di segnalarlo nella fase attuale.

Ringraziamo vivamente i Professori G. Bolla del C.I.S.E. per averci reso possibile l'impiego di un cristallo di  $\text{NaI(Tl)}$  particolarmente adatto per le nostre misurazioni, ed E. Perucca per il suo interessamento e per il suo aiuto in queste nostre ricerche.

#### SUMMARY

We deal with a differential discriminator where the pulses are conveniently displayed on the screen of a CRO, photographed and explored with a microphotometer. The discriminator is used in connection with a coincidence apparatus.

## A Sound-Wave Description of Fermion Assemblies.

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(ricevuto il 10 Novembre 1953)

**Summary.** — A description of assemblies of non-interacting fermions in terms of « sound-waves » is given. Additional degrees of freedom are needed for a complete determination of the system, but their contributions are expected to be small for all isotropic excitations. Special magnetic fields can also be included. The thermodynamic functions are calculated, and an application to the problem of density fluctuations is sketched.

### 1. — Introduction.

The need for a description of many-fermion problems in terms of collective motions has long been felt. In the theory of metals e.g. the interaction between electrons is often expected to be of long range. This invalidates the self-consistent field approach because the forces may well be such that a treatment of the residual interaction by perturbation methods fails. Typical examples of these difficulties appear in calculations of the Coulomb contribution to the free energy of an electron gas <sup>(1,2)</sup> and in the failure of perturbation methods <sup>(3,4)</sup> in Fröhlich's <sup>(5)</sup> theory of superconductivity.

(<sup>1</sup>) H. KOPPE: *Zeits. Naturforsch.*, **2a**, 429 (1947).

(<sup>2</sup>) J. BARDEEN: *Phys. Rev.*, **50**, 1098 (1936); E. P. WOHLFAHRT: *Phil. Mag.*, **41**, 534 (1950).

(<sup>3</sup>) M. R. SCHAFROTH: *Helv. Phys. Acta*, **24**, 645 (1951).

(<sup>4</sup>) G. WENTZEL: *Phys. Rev.*, **83**, 168 (1951); KUN HUANG: *Proc. Phys. Soc.*, **64**, 867 (1951).

(<sup>5</sup>) H. FRÖHLICH: *Phys. Rev.*, **79**, 845 (1950).

A way of arriving at a collective description of the many-fermion system which uses the concept of plasma-oscillations was developed by BOHM and PINES<sup>(6)</sup>. There the force ensuring a collective behaviour of the fermions is taken to be the Coulomb repulsion. As early as 1934, however, BLOCH<sup>(7)</sup> pointed out that another approach should be possible in which the correlating force between particles would be the repulsion due to the exclusion principle. The collective oscillations which are then expected to occur might well be called « sound-waves » in the fermion gas, since they are essentially conditioned by the zero-point pressure of the gas. This idea has proved very difficult to follow up. In fact, up to now only the one-dimensional case has been treated consistently by TOMONAGA<sup>(8)</sup>. FRÖHLICH<sup>(9)</sup> has recently pointed out that in the three-dimensional case one can define an assembly of bosons such that its thermodynamic behaviour at low temperature coincides with that of a Fermi gas.

In more than one dimension serious difficulties of topological nature are due to the fact that the Fermi surface of a fermion gas is connected, so that « resonance-transitions » along it are possible. This seems to forbid definitions of « sound-waves » whose amplitudes are bilinear in the fermion amplitudes and which give a complete description of the fermion assembly. On the other hand, it is sufficient to have a description where the « sound-waves » give the collective motions of the gas only, whereas some additional degrees of freedom are needed to describe the remaining random motions of the particles. In fact, such a splitting of the degrees of freedom occurs also in Bohm and Pines' plasma-oscillation treatment<sup>(6)</sup>.

The aim of the present paper is to give a possible construction of such « sound-waves » in an assembly of non-interacting fermions, where the main part of the motions is given by the « sound-waves », whereas additional degrees of freedom allow for motions of another kind; for many purposes these give only minor contributions. The disadvantage of the method is that its « sound-waves » have only a loose connection with what one would intuitively understand by that name.

We shall use a second-quantization formalism, where the assembly of fermions is described by quantized amplitudes  $a_s$ ,  $a_s^*$  of the fermion states  $|s\rangle$ , satisfying the usual anticommutation relations ( $\{A, B\} = AB + BA$ ):

$$(1) \quad \left\{ \begin{array}{l} \{a_s, a_{s'}\} = \{a_s^*, a_{s'}^*\} = 0, \\ \{a_s, a_{s'}^*\} = \delta_{ss'}. \end{array} \right.$$

<sup>(6)</sup> D. BOHM and D. PINES: I. *Phys. Rev.*, **82**, 625 (1951); II. *Phys. Rev.*, **85**, 338 (1952); III. in press.

<sup>(7)</sup> F. BLOCH: *Helv. Phys. Acta*, **7**, 385 (1934).

<sup>(8)</sup> S. TOMONAGA: *Progr. Theor. Phys.*, **5**, 544 (1950).

<sup>(9)</sup> H. FRÖHLICH: *Physica* **19**, 755 (1953).

We then quite generally call « sound-waves » any set of bilinear aggregates in the fermion amplitudes:

$$(2) \quad \left\{ \begin{array}{l} \varrho(\sigma) = \sum_{s, s'} C_{ss'}^{\sigma} a_s^* a_{s'} \\ \varrho^+(\sigma) = \sum_{s, s'} C_{ss'}^{+\sigma} a_s^* a_{s'} \end{array} \right. \quad (C_{ss'}^{+\sigma} = (C_{s's}^{\sigma})^*)$$

such that, in the approximation of low excitation of the fermion gas, they fulfil commutation relations of boson amplitudes ( $[A, B] = AB - BA$ ):

$$(3) \quad \left\{ \begin{array}{l} [\varrho(\sigma), \varrho(\sigma')] = [\varrho^+(\sigma), \varrho^+(\sigma')] = 0 \\ [\varrho(\sigma), \varrho^+(\sigma')] = \delta_{\sigma\sigma'} \end{array} \right.$$

and that their time dependence is given by

$$(4) \quad \left\{ \begin{array}{l} \dot{\varrho}(\sigma) = -i\omega(\sigma)\varrho(\sigma) \\ \dot{\varrho}^+(\sigma) = +i\omega(\sigma)\varrho^+(\sigma) \end{array} \right. \quad (\omega(\sigma) > 0).$$

Here  $A^* \equiv -(i\hbar)[H, A]$ ,  $H$  being the Hamiltonian of the system.

Clearly this definition embraces a much wider variety of motions than sound-waves in the usual sense. These would have to fulfil the additional requirement that the density operator  $\varrho(x) = |\int ds a_s \langle s|x \rangle|^2$  be expressible in terms of sound-waves alone. We keep the name of « sound-waves » for our more general definition (2) in order to have a simple nomenclature.

If, for any observable  $A$

$$(5) \quad [A, \varrho(\sigma)] = \lambda_{\sigma} \varrho(\sigma),$$

we say that the sound-waves « carry »  $A$ . (4) thus means that the  $\varrho(\sigma)$  carry a definite energy quantum  $\hbar\omega_{\sigma}$ ; they can, however, not be made to carry any other simple physical quantity (such as e.g. momentum, angular momentum...) simultaneously. In the construction we propose, the requirement that the sound-waves *carry* certain quantities is replaced by the requirement that they *conserve* certain quantities; i.e. we require the existence of some observable  $S$  which commute with  $H$ :  $[H, S] = 0$  and fulfill  $[\varrho(\sigma), S] = 0$ .

This allows a simple definition of sound-waves meeting all the requirements stated above. Furthermore, we shall show that the Hamiltonian of the free fermion gas can be split into a sound-wave part and a part involving some extra variables, the latter giving a negligible contribution to the thermodynamic functions at low temperatures. BLOCH's (?) result for the density fluctuations can easily be derived, and a magnetic field can be included.



On the other hand, it is obvious that such a definition of sound-waves has serious shortcomings. Any interaction will induce transitions in some physical quantities of the gas, and the fact that the sound-waves cannot carry this quantity shows that transitions between collective and random motions will be essential. We cannot hope therefore to simplify interactions as radically as in the one-dimensional case <sup>(8)</sup>. It should be mentioned, however, that the one-dimensional model <sup>(8)</sup> of TOMONAGA is a special case of our construction, the quantity  $S$  which is conserved being the only one needed beside the energy in order to determine the states of a one-dimensional particle: i.e. the direction of momentum. For this reason, already in one dimension interactions which combine a large transfer of momentum with only small changes in energy as induced e.g. by fields of wave number near  $2(2m\zeta/\hbar^2)^{1/2}$  ( $\zeta$  = Fermi energy,  $m$  = electron mass) cannot be easily included. However, here this difficulty may be avoided by considering only long-range-forces, whereas in more dimensions no such restriction is helpful. This stems from the fact that the degeneracy of a state of a fermion with given energy  $E$  is independent of  $E$  only in the one-dimensional case; for higher dimensions ( $n$ ) the degeneracy increases as  $M(E)^{(n-1)/n}$ , where  $M(E)$  is the number of states with energy  $\leq E$ . This is an equivalent statement of the aforementioned topological difficulty.

## 2. - A sound-wave description of an assembly of non-interacting fermions.

We consider an assembly of  $N$  non-interacting fermions described by a quantized wave-function  $\psi(x)$ , with

$$(6) \quad \begin{cases} \{\psi(x), \psi(x')\} = \{\psi^*(x), \psi^*(x')\} = 0, \\ \{\psi(x), \psi^*(x')\} = \delta(x - x'). \end{cases}$$

The Hamiltonian is

$$(7) \quad \begin{aligned} H &= \int dx \psi^*(x) \left( -\frac{\hbar^2}{2m} \Delta + V(x) \right) \psi(x) \equiv \int dx \psi^*(x) h \psi(x) \\ h &\equiv -\frac{\hbar^2}{2m} \Delta + V(x). \end{aligned}$$

(Throughout this paragraph, we do not restrict ourselves to a definite number of spatial dimensions; we write  $dx$  for the volume element).

We suppose  $V(x)$  to be such that the spectrum of  $h$  is discrete, or, alternatively, we use a suitable periodicity volume. We choose an arbitrary

complete set of eigenstates  $|\eta s\rangle$ , where  $\eta$ ,  $s$  are some quantum numbers. Without loss of generality we may assume  $\eta$  to be capable of all positive integer values, whereas  $s$  runs over a certain discrete set  $\mathcal{S}$ . Expanding

$$(8) \quad \begin{cases} \psi(x) = \sum_{\eta s} a_{\eta s} \langle x | \eta s \rangle \\ \psi^*(x) = \sum_{\eta s} a_{\eta s}^* \langle \eta s | x \rangle \end{cases}$$

the  $a_{\eta s}$  satisfy the relations

$$(1') \quad \begin{cases} \{a_{\eta s}, a_{\eta' s'}\} = \{a_{\eta s}^*, a_{\eta' s'}^*\} = 0 \\ \{a_{\eta s}, a_{\eta' s'}^*\} = \delta_{\eta\eta'} \delta_{ss'} \end{cases}$$

$\mathcal{N}_{\eta s} = a_{\eta s}^* a_{\eta s}$  are the occupation number operators with eigenvalues 0, 1.

The Hamiltonian becomes

$$(9) \quad H = \sum_{\eta s} E_{\eta s} a_{\eta s}^* a_{\eta s},$$

where  $h|\eta s\rangle = E_{\eta s}|\eta s\rangle$ .

We suppose  $E_{\eta s}$  to be monotonically increasing with  $\eta$  for fixed  $s$ . The lowest state of the assembly is given by

$$\begin{aligned} \mathcal{N}_{\eta s} &= 1 & \text{for} & & E_{\eta s} \leq \zeta \\ \mathcal{N}_{\eta s} &= 0 & \text{for} & & E_{\eta s} > \zeta. \end{aligned}$$

$\zeta$  is the Fermi energy.

We now define sound-waves with the properties stated in § 1:

$$(10) \quad \begin{cases} \varrho_{\varepsilon s} = \sum_{\eta} a_{\eta s}^* a_{\eta+\varepsilon, s} \\ \varrho_{\varepsilon s}^+ = \sum_{\eta} a_{\eta+\varepsilon, s}^* a_{\eta s} \end{cases} \quad (\varepsilon > 0)$$

or more concisely:

$$(11) \quad \begin{cases} \varrho_{\varepsilon s} = \sum_{\eta} a_{\eta s}^* a_{\eta+\varepsilon, s} \theta(\eta + \varepsilon) \\ \varrho_{\varepsilon s}^+ = \varrho_{-\varepsilon, s} \end{cases} \quad (\varepsilon \geq 0)$$

with

$$(12) \quad \theta(x) = \begin{cases} 1 & \text{for } x \geq 0 \\ 0 & \text{for } x < 0. \end{cases}$$

One sees that they « conserve » (in the sense of § 1) the quantities  $S$  labelled by the quantum numbers  $s$ .

The commutator of the  $\varrho$ -s is:

$$\begin{aligned}
 (\varepsilon > \varepsilon'): \quad [\varrho_{\varepsilon s}, \varrho_{\varepsilon' s'}] &= \sum_{\eta \eta'} \{ a_{\eta s}^* \delta(\eta + \varepsilon, \eta') \delta_{ss'} a_{\eta' + \varepsilon', s} \theta(\eta + \varepsilon) \theta(\eta' + \varepsilon') - \\
 &\quad - a_{\eta' s'}^* \delta(\eta' + \varepsilon', \eta) \delta_{ss'} a_{\eta + \varepsilon, s} \theta(\eta + \varepsilon) \theta(\eta' + \varepsilon') \} = \\
 &= \sum_{\eta} a_{\eta s}^* a_{\eta + \varepsilon + \varepsilon', s} (\theta(\eta + \varepsilon) - \theta(\eta + \varepsilon')) \theta(\eta + \varepsilon + \varepsilon').
 \end{aligned}$$

For  $\varepsilon > \varepsilon' > 0$  this vanishes because  $\theta(\eta + \varepsilon) = \theta(\eta + \varepsilon') = 1$ ; for  $0 > \varepsilon > \varepsilon'$  it vanishes because either  $\theta(\eta + \varepsilon) = \theta(\eta + \varepsilon') = 1$  or  $\theta(\eta + \varepsilon + \varepsilon') = 0$ . For  $\varepsilon > 0 > \varepsilon'$  it becomes

$$(13) \quad [\varrho_{\varepsilon s}, \varrho_{\varepsilon' s'}] = \sum_{\eta=0}^{|\varepsilon'|} a_{\eta s}^* a_{\eta + \varepsilon + \varepsilon', s} \theta(\eta + \varepsilon + \varepsilon').$$

Thus we have, for  $\varepsilon > 0, \varepsilon' > 0$ :

$$(14) \quad \begin{cases} [\varrho_{\varepsilon s}, \varrho_{\varepsilon' s'}] = [\varrho_{\varepsilon s}^+, \varrho_{\varepsilon' s'}^+] = 0, \\ [\varrho_{\varepsilon s}, \varrho_{\varepsilon' s'}^+] = \delta_{ss'} \sum_{\eta=0}^{\varepsilon'} a_{\eta s}^* a_{\eta + \varepsilon - \varepsilon', s} \theta(\eta + \varepsilon - \varepsilon'). \end{cases}$$

(It should be noted that we are consistently neglecting terms of the relative order  $0(1/N)$ ; in this sense in the above expressions we have not been careful in treating the limits of summation which in the sum  $\sum_{\eta}$  should strictly be  $\varepsilon' - 1$  instead of  $\varepsilon'$ ).

We now make the following assumptions which will enable (13) to be simplified (see fig. 1):

- I) All fermion states  $|\eta s\rangle$  with  $E_{\eta s} < \zeta(1 - \lambda_0)$ , ( $\lambda_0 \ll 1$ ) are filled, i.e.  $\mathcal{N}_{\eta s} = 1$ ;  
all fermions states  $|\eta s\rangle$  with  $E_{\eta s} > \zeta(1 + \lambda_0)$ ;  
are empty ( $\mathcal{N}_{\eta s} = 0$ ).

- II)  $\varepsilon < \varepsilon_1 \ll \eta_0$ , where

$$\eta_0 = \text{Max } \eta_s \text{ over all } s$$

and  $\eta_s$  is defined as being the quantum number giving the Fermi limit for fixed  $s$ :  $E_{\eta_s s} = \zeta$ .

III)  $s$  in (10) shall be restricted to a certain set  $S$  such that for all  $s \in S$ :  $\eta_s \gg \varepsilon_1$  and  $E_{0s} < \zeta(1 - \lambda_0)$ .

For states with  $s \in S$  I) can be replaced by

$$\text{I')} \quad \begin{cases} \mathcal{N}_{\eta s} = 1 & \text{if } s \in S, \quad \eta < \eta_s(1 - \lambda_1) \\ \mathcal{N}_{\eta s} = 0 & \text{if } s \in S, \quad \eta > \eta_s(1 + \lambda_1) \end{cases} \quad (\lambda_1 \ll 1)$$

provided the variation of the energy function with  $\eta$  at fixed  $s$  has a uniform upper bound over  $S$ .

We can now conclude from II) that for  $s \in S$  the states  $|\eta s\rangle$  and  $|\eta + \varepsilon - \varepsilon', s\rangle \equiv |\eta', s\rangle$  appearing in (13) fulfil  $\eta, \eta' < \varepsilon' < \varepsilon_1 \ll \eta_s$  and I') then states that these states are filled.

As  $a_{\eta s}^*$  applied to a filled state vanishes,  $a_{\eta s}^* a_{\eta' s}$  is different from zero only for  $\eta = \eta'$ , being then equal to  $\mathcal{N}_{\eta s} = 1$ .

So we get finally ( $\varepsilon, \varepsilon' > 0$ )

$$(15) \quad \begin{cases} [\varrho_{\varepsilon s}, \varrho_{\varepsilon' s'}] = [\varrho_{\varepsilon s}^+, \varrho_{\varepsilon' s'}^+] = 0 \\ [\varrho_{\varepsilon s}, \varrho_{\varepsilon' s'}^+] = \delta_{ss'} \delta_{\varepsilon \varepsilon'} \varepsilon \end{cases}$$

under the conditions I), II), III).

Thus up to a factor our  $\varrho$ , (10), have the required property (3).

Our next task is to express the Hamiltonian (9) in terms of the sound-waves (10). We write

$$(16) \quad \begin{cases} H = \sum_{\substack{\eta s \\ s \in S}} E_{\eta s} a_{\eta s}^* a_{\eta s} + H'; & H' \equiv \sum_{\substack{\eta s \\ s \notin S}} E_{\eta s} a_{\eta s}^* a_{\eta s} \\ H - H' = \sum_{\substack{\eta s \\ s \in S \\ E_{\eta s} < \zeta}} E_{\eta s} + \sum_{\substack{\eta s \\ s \in S \\ E_{\eta s} < \zeta}} E_{\eta s} (a_{\eta s}^* a_{\eta s} - 1) + \sum_{\substack{\eta s \\ s \in S \\ E_{\eta s} > \zeta}} E_{\eta s} a_{\eta s}^* a_{\eta s}, \end{cases}$$

$$(17) \quad \sum_{\substack{E_{\eta s} < \zeta \\ s \in S}} E_{\eta s} \equiv E_0(S) \text{ is a constant zero-point energy.}$$

In the other two terms condition I') ensures that only states near the Fermi surface contribute. We may therefore expand  $E_{\eta s}$  around  $\eta_s$ :

$$(18) \quad E_{\eta s} = \zeta + (\eta - \eta_s)\varphi(s) + \dots$$

and drop quadratic terms; the error will be of relative order  $\lambda_0$ .

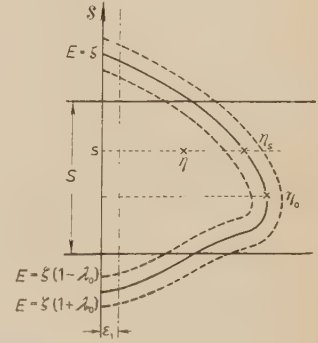


Fig. 1. — Fermi distribution and sound-wave variables in  $(\eta s)$ -space.



We then obtain

$$(19) \quad H = H' + E_0(S) + \sum_{s \in S} (\mathcal{N}_s - \eta_s) \zeta + \sum_{\substack{s \in S \\ \eta}} (\eta - \eta_s) \varphi(s) \mathcal{N}_{\eta_s} - \sum_{\substack{s \in S \\ 0 < \eta < \eta_s}} (\eta - \eta_s) \varphi(s),$$

where

$$\mathcal{N}_s \equiv \sum_{0 \leq \eta < \infty} \mathcal{N}_{\eta s}.$$

We now make use of the equality:

$$(20) \quad \sum_{\varepsilon} \varrho_{\varepsilon s}^+ \varrho_{\varepsilon s} = \sum_{\eta} \eta a_{\eta s}^* a_{\eta s} - 1/2 \mathcal{N}_s^2$$

which will be proved later. This allows (19) to be written

$$(21) \quad H = H' + \sum_{\substack{s \in S \\ \varepsilon}} \varphi(s) \varrho_{\varepsilon s}^+ \varrho_{\varepsilon s} + 1/2 \sum_{s \in S} \varphi(s) (\mathcal{N}_s^2 - 2\eta_s \mathcal{N}_s) + \\ + \zeta \sum_{s \in S} (\mathcal{N}_s - \eta_s) + E_0(S) - \sum_{\substack{s \in S \\ 0 < \eta < \eta_s}} (\eta - \eta_s) \varphi(s).$$

The last term is

$$\sum_{\substack{s \in S \\ 0 < \eta < \eta_s}} (\eta - \eta_s) \varphi(s) = \sum_{s \in S} 1/2 \eta_s^2 \varphi(s).$$

We introduce new variables

$$(22) \quad \Delta_s \equiv \mathcal{N}_s - \eta_s$$

which, strictly speaking, range over all integers between  $-\eta_s$  and  $+\infty$ ; actually by conditions I) II) we may allow  $\Delta_s$  to take all integer values.

We then get

$$(23) \quad H = E_0(S) + \sum_{\substack{s \in S \\ \eta}} \varphi(s) \varrho_{\varepsilon s}^+ \varrho_{\varepsilon s} + 1/2 \sum_{s \in S} \varphi(s) \Delta_s^2 + \zeta \sum_{s \in S} \Delta_s + \sum_{\substack{\eta_s \\ s \notin S}} E_{\eta_s} \mathcal{N}_{\eta_s}.$$

We see therefore that the Fermi gas is completely described by the variables

$$N_{\varepsilon s} \equiv \frac{1}{\varepsilon} \varrho_{\varepsilon s}^+ \varrho_{\varepsilon s} = \text{occupation number of sound-wave bosons},$$

$$\mathcal{N}_{\eta_s} = \text{occupation number of « outside » electrons } (s \notin S),$$

$$\Delta_s \quad \text{given by (22).}$$

The latter two are subject to the subsidiary condition

$$\sum_{s \in S} \mathcal{N}_s + \sum_{\substack{s \notin S \\ \eta}} \mathcal{N}_{\eta s} = N$$

or

$$(24) \quad \sum_{s \in S} \Delta_s + \sum_{\substack{s \notin S \\ \eta}} \mathcal{N}_{\eta s} = 0.$$

Using (24) we can simplify (23) once more and get:

$$(23') \quad H = E_0(S) + \sum_{s \in S} \varphi(s) \varrho_{es}^+ \varrho_{es} + \frac{1}{2} \sum_{s \in S} \varphi(s) \Delta_s^2 + \sum_{\substack{\eta, s \notin S}} (E_{\eta s} - \zeta) \mathcal{N}_{\eta s}.$$

The Hamiltonian (23') thus splits into three parts:

$$(25) \quad H = H_0 + H_B + H_C$$

with  $H_0 = E_0(S)$ , a constant zero-point energy

$$H_B = \sum_{\varepsilon} \sum_{s \in S} \varepsilon \varphi(s) N_{\varepsilon s}, \quad \text{the sound-wave energy,}$$

$$H_C = \frac{1}{2} \sum_{s \in S} \varphi(s) \Delta_s^2 + \sum_{\substack{\eta \\ s \notin S}} (E_{\eta s} - \zeta) \mathcal{N}_{\eta s}.$$

It will be shown in the next section that in the case of an ideal gas  $H_C$  is negligible for thermal excitations at low temperatures, so that one can reasonably expect  $H_C$  to be small in all cases and for all low excitations of random nature.

We give now the proof of (20) (\*). Writing out the left-hand side, using (10) we obtain

$$\sum_{\varepsilon > 0} \varrho_{\varepsilon s}^+ \varrho_{\varepsilon s} = \sum_{\varepsilon > 0} \sum_{\eta \eta'} a_{\eta + \varepsilon, s}^* a_{\eta, s} a_{\eta', s}^* a_{\eta' + \varepsilon, s}.$$

We discuss the terms  $\eta = \eta'$  and  $\eta \neq \eta'$  separately:

$$\eta = \eta': \quad \sum_{\varepsilon} \sum_{\eta} a_{\eta + \varepsilon, s}^* a_{\eta, s} a_{\eta, s}^* a_{\eta + \varepsilon, s} = \sum_{\varepsilon} \sum_{\eta} \mathcal{N}_{\eta + \varepsilon, s} (1 - \mathcal{N}_{\eta, s}).$$

Now:

$$\sum_{\varepsilon > 0} \sum_{\eta} \mathcal{N}_{\eta + \varepsilon, s} = \sum_{\eta' > 0} \sum_{\varepsilon = 0}^{\eta'} \mathcal{N}_{\eta' s} = \sum_{\eta' > 0} \eta' \mathcal{N}_{\eta' s}$$

(\*) This proof is identical with the one given by TOMONAGA for the equivalent statement in the one-dimensional case (8).

and

$$\sum_{\varepsilon > 0} \sum_{\eta} \mathcal{N}_{\eta+\varepsilon, s} \mathcal{N}_{\eta s} = \sum_{\eta} \sum_{\eta' > \eta} \mathcal{N}_{\eta' s} \mathcal{N}_{\eta s} = \frac{1}{2} (\sum_{\eta} \mathcal{N}_{\eta s})^2$$

so that the terms with  $\eta = \eta'$  give the right-hand side of (20).

The terms  $\eta \neq \eta'$  vanish: Using (1') we get

$$A \equiv \sum_{\varepsilon > 0} \sum_{\eta' > 0} \sum_{\eta > \eta'} a_{\eta+\varepsilon}^* a_{\eta} a_{\eta'}^* a_{\eta'+\varepsilon} = - \sum_{\varepsilon > 0} \sum_{\eta' > 0} \sum_{\eta > \eta'} a_{\eta+\varepsilon}^* a_{\eta'+\varepsilon} a_{\eta'}^* a_{\eta}.$$

Defining now

$$\bar{\varepsilon} = \eta - \eta'$$

$$\bar{\eta} = \eta' + \varepsilon$$

$$\bar{\eta}' = \eta'$$

this become

$$- \sum_{\bar{\varepsilon} > 0} \sum_{\bar{\eta}' > 0} \sum_{\bar{\eta} > \bar{\eta}'} a_{\bar{\eta}+\bar{\varepsilon}}^* a_{\bar{\eta}} a_{\bar{\eta}'}^* a_{\bar{\eta}'+\bar{\varepsilon}'} = -A$$

which implies  $A=0$ . Similarly one proves

$$\sum_{\varepsilon > 0} \sum_{\eta' > 0} \sum_{\eta < \eta'} a_{\eta+\varepsilon}^* a_{\eta} a_{\eta'}^* a_{\eta'+\varepsilon} = 0$$

which completes the proof of (20).

Finally we wish to point out that in the case of a one-dimensional free fermion gas our method reduces to TOMONAGA's (8). Take the wave functions of the fermions in a box of length  $L$  to be

$$\langle x | \eta s \rangle = L^{-1/2} \exp \left[ - \frac{2\pi i}{L} s \eta x \right].$$

$\eta$  runs over all positive integers,  $s$  over  $\pm 1$ . Then our definition (11) of  $\varrho_{\varepsilon s}$  is equivalent to Tomonaga's except for some transitions of fermions from small negative to small positive momenta (and vice-versa) which are absent in (11), whereas they are included by TOMONAGA. As, however, in view of I) these never contribute, this difference is of no importance.

No restriction on the  $s$  is needed, so that  $S = \mathcal{S}$ , and no variables  $\mathcal{N}_{\eta s}$  occur. There are two quantities  $\Delta_s$ ,  $\Delta_+$  and  $\Delta_-$ , related by

$$(24') \quad \Delta_+ + \Delta_- = 0.$$

Denoting then  $\Delta_+ = -\Delta_- \equiv \Delta$  the Hamiltonian becomes

$$(23'') \quad H = E_0 + \left( \frac{2\pi\hbar}{L} \right)^2 \frac{n_0}{m} \left[ \sum_{\epsilon > 0} (\varrho_{\epsilon+}^+ \varrho_{\epsilon+} + \varrho_{\epsilon-}^+ \varrho_{\epsilon-}) + \Delta^2 \right].$$

This is Tomonaga's expression except for the  $\Delta^2$ -term which accounts for the possibility that the number of electrons with positive momenta may differ from the number of electrons with negative momenta, a feature neglected by TOMONAGA.

### 3. - Application to the ideal Fermi gas.

A) *Generalities.* - As a special case we consider a 3-dimensional ideal gas of  $N$  spinless fermions in a periodicity cube of volume  $L^3$ . The one-particle states are  $|n_1, n_2, n_3\rangle \equiv |\mathbf{n}\rangle$  where  $(n_1, n_2, n_3) = \mathbf{n}$  are quantum numbers assuming all integer values. The state  $|\mathbf{n}\rangle$  has a definite momentum

$$(26) \quad \mathbf{p}_{\mathbf{n}} = \frac{2\pi\hbar}{L} \mathbf{n}$$

and energy

$$(27) \quad E_{\mathbf{n}} = \left( \frac{2\pi\hbar}{L} \right)^2 \frac{\mathbf{n}^2}{2m}.$$

We write

$$(28) \quad \zeta = \left( \frac{2\pi\hbar}{L} \right)^2 \frac{n_0^2}{2m}$$

and

$$(28') \quad N = \frac{4\pi}{3} n_0^3,$$

We choose

$$(29) \quad \eta = |\mathbf{n}_1| \quad \text{and} \quad s = \left( n_2, n_3; \sigma \equiv \frac{n_1}{|\mathbf{n}_1|} \right).$$

(This introduces a preference for the  $x$ -axis in space).

(10) then becomes:

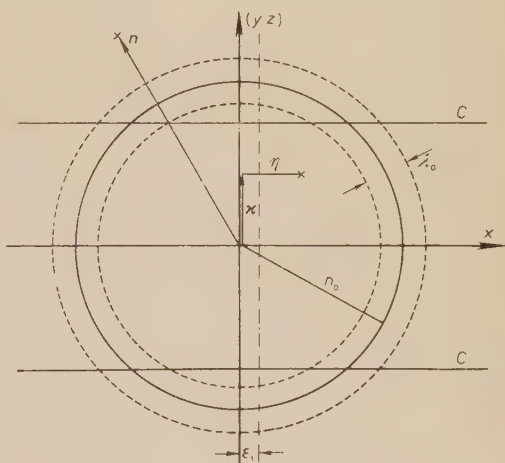


Fig. 2. - Fermi distribution and sound-wave variables in momentum space.  $C$  = cut-off cylinder  $\kappa = n_0^2 (1 - \gamma)$ .



(we denote the two-dimensional lattice vector  $(n_2, n_3)$  by  $\mathbf{x}$ )

$$\varrho_{\varepsilon\mathbf{x}+} = \sum_{\substack{n_1 \geq 0 \\ (n_2, n_3) = \mathbf{x}}} a_{n_1, n_2, n_3}^* a_{n_1 + \varepsilon, n_2, n_3} \quad (\varepsilon > 0)$$

$$\varrho_{\varepsilon\mathbf{x}-} = \sum_{\substack{n_1 \leq 0 \\ (n_2, n_3) = \mathbf{x}}} a_{n_1, n_2, n_3}^* a_{n_1 - \varepsilon, n_2, n_3} \quad (\varepsilon > 0)$$

$$\varrho_{\varepsilon\mathbf{x}+}^+ = \sum_{\substack{n_1 \geq 0 \\ (n_2, n_3) = \mathbf{x}}} a_{n_1 + \varepsilon, n_2, n_3}^* a_{n_1, n_2, n_3} \quad (\varepsilon > 0)$$

$$\varrho_{\varepsilon\mathbf{x}-}^+ = \sum_{\substack{n_1 \leq 0 \\ (n_2, n_3) = \mathbf{x}}} a_{n_1 - \varepsilon, n_2, n_3}^* a_{n_1, n_2, n_3}; \quad (\varepsilon > 0)$$

(18) becomes, with

$$(31) \quad \eta_s \equiv \eta_{\mathbf{x}\pm} = (n_0^2 - \mathbf{x}^2)^{1/2}$$

$$(32) \quad E_{\eta_s} = \zeta + (n_1 - \eta_s) \frac{2\zeta}{n_0^2} (n_0^2 - \mathbf{x}^2)^{1/2}$$

so that

$$(33) \quad \varphi(s) = \frac{2\zeta}{n_0^2} (n - \mathbf{x}^2)^{1/2} = \varphi(\mathbf{x}).$$

The set  $S$  will be defined by

$$\mathbf{x}^2 \equiv n_2^2 + n_3^2 \leq n_0^2(1 - \gamma). \quad (\gamma \ll 1).$$

For given  $\lambda_0 \ll 1$  and  $\varepsilon_1 \ll n_0$  III) then requires

$$(34) \quad n_0^2 \gamma \gg \varepsilon_1,$$

$$(35) \quad \gamma \gg \lambda_0.$$

The equivalence of I') with I) is fulfilled if we take

$$(36) \quad \lambda_1 = \frac{\lambda_0}{2\gamma} \ll 1$$

which is consistent as follows from (35).

We now simply take over the commutation relation and the Hamiltonian from § 3. One has, for  $\varepsilon > 0$ :

$$(37) \quad \left\{ \begin{array}{l} [\varrho_{\varepsilon\mathbf{x}\sigma}, \varrho_{\varepsilon'\mathbf{x}'\sigma'}] = 0 \\ [\varrho_{\varepsilon\mathbf{x}\sigma}^+, \varrho_{\varepsilon'\mathbf{x}'\sigma'}^+] = 0 \\ [\varrho_{\varepsilon\mathbf{x}\sigma}, \varrho_{\varepsilon'\mathbf{x}'\sigma'}^+] = \varepsilon \delta_{\varepsilon\varepsilon'} \delta_{\mathbf{x}\mathbf{x}'} \delta_{\sigma\sigma'} \end{array} \right. \quad (\sigma, \sigma' = \pm 1)$$

and

$$(38) \quad \left\{ \begin{array}{l} H_0 = \frac{3}{5} N \zeta(1 - 0(\gamma)) \quad (\text{the zero-point energy}) \\ H_B = \sum_{\varepsilon > 0} \sum_{|\mathbf{x}| < n_0(1 - \gamma/2)} \varphi(\mathbf{x}) (\varrho_{\varepsilon+}^+ + \varrho_{\varepsilon\mathbf{x}+} + \varrho_{\varepsilon-}^+ - \varrho_{\varepsilon\mathbf{x}-}) \\ \quad = \sum_{\varepsilon > 0} \sum_{\sigma = \pm 1} \sum_{|\mathbf{x}| < n_0(1 - \gamma/2)} \varepsilon \varphi(\mathbf{x}) N_{\varepsilon\mathbf{x}\sigma} \\ H_C = \frac{1}{2} \sum_{\sigma} \sum_{|\mathbf{x}| < n_0(1 - \gamma/2)} \varphi(\mathbf{x}) \Delta_{\sigma}^2 + \sum_{\mathbf{n}} (E_{\mathbf{n}} - \zeta) \mathcal{N}_{\mathbf{n}} \end{array} \right.$$

$n_0^2 + n_0^2 > n_0^2(1 - \gamma)$

with

$$(39) \quad \mathcal{J}(\Delta, \mathcal{N}) \equiv \sum_{\sigma} \sum_{|\mathbf{x}| < n_0(1 - \gamma/2)} \Delta_{\sigma} + \sum_{\mathbf{n}} \mathcal{N}_{\mathbf{n}} = 0,$$

$n_0^2 + n_0^2 > n_0^2(1 - \gamma)$

B) *Thermodynamics.* — The free energy  $F$  of the gas splits into three parts in the same way as the Hamiltonian:

$$(40) \quad F = F_0 + F_B + F_C,$$

$F_0 = H_0$ .  $F_B$  is calculated by standard methods to give

$$F_B = + kT \sum_{\sigma, \varepsilon, |\mathbf{x}| < n_0(1 - \gamma/2)} \log(1 - \exp[-\alpha \varepsilon \varphi(\mathbf{x})]); \quad \alpha \equiv 1/kT.$$

Replacing sums by integrals, (which means taking  $N \gg \alpha \zeta$ ) gives

$$\begin{aligned} F_B &= 2kT \int_0^\infty d\varepsilon \int_0^{n_0(1 - \gamma/2)} d^2\mathbf{x} \log \left( 1 - \exp \left[ -\frac{2\alpha\zeta}{n_0^2} \sqrt{n_0^2 - \mathbf{x}^2} \right] \right) = \\ &= \frac{n_0^2}{\zeta} 2(kT)^2 \int_0^{n_0(1 - \gamma/2)} \frac{d^2\mathbf{x}}{2\sqrt{n_0^2 - \mathbf{x}^2}} \int_0^\infty dx \log(1 - e^{-x}) = \\ &= \frac{n_0^2}{\zeta} (kT)^2 \cdot 2\pi n_0(1 - \sqrt{\gamma}) \left( -\frac{\pi^2}{6} \right), \end{aligned}$$

$$(41) \quad F_B = -\frac{\pi^2}{4} \frac{N(kT)^2}{\zeta} (1 - \sqrt{\gamma}) \quad (\gamma \ll 1).$$

The evaluation of  $F_C$  requires consideration of the subsidiary condition (39). In order to deal with it we require its fulfilment only on a thermal average, i.e. we use a grand canonical ensemble. Introducing a chemical potential  $\mu$ , we write

$$(42) \quad \exp[-\alpha F_C(\mu)] = \sum_{(\Delta), (\mathcal{N})} \exp[-\alpha H_C(\Delta, \mathcal{N}) + \alpha \mu \mathcal{J}(\Delta, \mathcal{N})]$$

and require

$$(43) \quad \frac{\partial F_c}{\partial \mu} = 0$$

to determine  $\mu$ .

$F_c(\mu)$  then splits into two parts, one involving only the  $\Delta$ , the other only the  $\mathcal{Q}$ :

a) The part containing the  $\Delta$ ,  $F_{c1}$ , is given by

$$\begin{aligned} \exp[-\alpha F_{c1}] &= \sum_{(\Delta_{\mathbf{x}\sigma})} \exp \left[ -\alpha \sum_{\sigma, \mathbf{x} \in S} \Delta_{\mathbf{x}\sigma}^2 \cdot \frac{1}{2} \varphi(\mathbf{x}) + \alpha \mu \sum_{\sigma, \mathbf{x} \in S} \Delta_{\mathbf{x}\sigma} \right] \\ &= \prod_{\sigma, \mathbf{x} \in S} \left\{ \sum_{\Delta_{\mathbf{x}\sigma} = -\infty}^{+\infty} \exp \left[ -\frac{\alpha}{2} \varphi(\mathbf{x}) \left( \Delta_{\mathbf{x}\sigma}^2 - \frac{2\mu}{\varphi(\mathbf{x})} \Delta_{\mathbf{x}\sigma} \right) \right] \right\} \\ &= \prod_{\sigma, \mathbf{x} \in S} \left\{ \sum_{\Delta_{\mathbf{x}\sigma} = -\infty}^{+\infty} \exp \left[ -\frac{\alpha}{2} \varphi(\mathbf{x}) \left( \Delta_{\mathbf{x}\sigma} - \frac{\mu}{\varphi(\mathbf{x})} \right)^2 \right] \cdot \exp \left[ \alpha \frac{\mu^2}{2\varphi(\mathbf{x})} \right] \right\}. \end{aligned}$$

The replacement of the sum over  $\Delta_{\mathbf{x}\sigma}$  by an integral is again seen to be permissible for  $\alpha \zeta \ll L$ , and we get

$$\begin{aligned} F_{c1} &= -\frac{\mu^2}{4} \sum_{\mathbf{x}\sigma} (n_0^2 - \mathbf{x}^2)^{-1/2} \frac{n_0^2}{\zeta} \left[ 1 + O\left(\frac{\log N}{N}\right) \right] = -\frac{\mu^2}{2} \frac{n_0^2}{\zeta} \cdot 2\pi n_0 (1 - \sqrt{\gamma}), \\ (44) \quad F_{c1} &= -\frac{3\mu^2}{4} \frac{N}{\zeta} (1 - \sqrt{\gamma}). \end{aligned}$$

b) The part containing the  $\mathcal{Q}$ :

$$F_{c2} = -kT \sum_{\substack{\mathbf{n} \\ n_2^2 + n_3^2 > n_0^2(1-\gamma)}} \log \left( 1 + \exp \left[ -\alpha \frac{\zeta}{n_0^2} (\mathbf{n}^2 - n_0^2) + \alpha \mu \right] \right),$$

becomes upon introducing  $t \equiv \zeta((\mathbf{n}^2/n_0^2) - 1)$ :

$$(45) \quad F_{c2} = -\frac{3}{2} N \frac{kT}{\zeta^{3/2}} \int_0^\infty dt \sqrt{t} \log(1 + \exp[-\alpha(t - \mu - n_0^2 \gamma)]).$$

In order to avoid a lengthy mathematical discussion of the dependence of (41), (44) and (45) upon  $\gamma$  and of the best choice for  $\gamma$  we shall restrict ourselves to the evaluation of the lowest order terms of  $F$  in  $kT$ . We then note

that for a low temperature  $T$ ,  $\lambda_0$  will be of the order  $kT/\zeta$ , so that we can take  $\gamma$  to vary with  $kT/\zeta$ . In order to obtain the lowest-order terms of  $F$ , we may therefore neglect  $\gamma$  altogether. This is easily seen to produce an error of the relative order  $(kT/\zeta)^{1/2}$ .

In this lowest approximation we find:

$$(46) \quad \left\{ \begin{aligned} F_B &= -\frac{\pi^2}{4} \frac{N}{\zeta} (kT)^2, \\ F_C &= -\frac{3}{4} N \frac{\mu^2}{\zeta} - \frac{3}{2} N \frac{kT}{\zeta^{3/2}} \int_0^\infty dt t^{1/2} \log(1 + \exp[\alpha(t - \mu)]), \end{aligned} \right.$$

(43) becomes:

$$(43') \quad \frac{\partial F_C}{\partial \mu} = -\frac{3}{2} N \frac{\mu}{\zeta} - \frac{3}{2} N \frac{kT}{\zeta^{3/2}} \int_0^\infty dt t^{1/2} (\exp[\alpha(t - \mu)] + 1)^{-1} = 0.$$

One expects  $\mu/kT$  to turn out  $\ll 1$ , so we may replace the integral by its value for  $\alpha\mu = 0$ :

$$\int_0^\infty dt t^{1/2} (\exp[\alpha(t - \mu)] + 1)^{-1} \rightarrow (kT)^{3/2} \int_0^\infty dx x^{1/2} (e^x + 1)^{-1} = \beta (kT)^{3/2},$$

$$\beta = \frac{1}{2} \sqrt{\pi} \frac{\sqrt{2} - 1}{\sqrt{2}} \approx \left(\frac{3}{2}\right),$$

( $\zeta(3/2)$  being the Riemann  $\zeta$ -function)

So we get:

$$(43'') \quad \mu = -\beta \zeta \left( \frac{kT}{\zeta} \right)^{3/2}.$$

Inserting this into (46) we find:

$$(47) \quad F_C = -\frac{3}{4} N \beta^2 \left( \frac{kT}{\zeta} \right)^2 kT - \frac{3}{2} N \beta' \left( \frac{kT}{\zeta} \right)^{3/2} kT,$$

with

$$\beta' = \frac{1}{2} \sqrt{\pi} \frac{\sqrt{2} - 1/2}{\sqrt{2}} \approx \left(\frac{5}{2}\right).$$

We thus see that up to terms of the relative order  $(kT/\zeta)^{1/2}$  the sound-wave excitations alone contribute to the thermal behaviour of the fermion gas (\*). The relatively low order of the error,  $(kT/\zeta)^{1/2}$ , stems from our crude choice of  $\gamma$  and could probably be improved. A somewhat better approach is also provided by choosing another representation of the fermion gas, e.g. in angular momentum space, where the results are less sensitive on the choice of  $\gamma$ . The special sensitivity of the momentum space representation on  $\gamma$ , i.e. the fact that the errors in  $F$  are  $O(\sqrt{\gamma})$ , stems from the fact that the cut-off cylinder  $\mathbf{x}^2 = n_0^2(1 - \gamma)$  cuts the Fermi surface under an angle tending towards zero with  $\gamma$ ; this is avoided in angular momentum space, and as a consequence the contributions of  $F_c$  are smaller than  $F_B$  by a factor  $(kT/\zeta)^{1/2}$ .

C) *Magnetic field.* — The action of a magnetic field can readily be included, provided we may restrict ourselves to fields dependent on the  $x$ -coordinate only. These are always sufficient to calculate the magnetic properties of the gas by perturbation methods <sup>(3,10)</sup>.

For simplicity we consider the case of a periodic field with vector potential

$$\mathbf{A}(\mathbf{x}) = \mathbf{A} \exp \left[ \frac{2\pi i}{L} \boldsymbol{\lambda}_0 \cdot \mathbf{x} \right] + \mathbf{A}^* \exp \left[ + \frac{2\pi i}{L} \boldsymbol{\lambda}_0 \cdot \mathbf{x} \right],$$

$$\boldsymbol{\lambda}_0 \equiv (\varepsilon_0, 0, 0).$$

This gives rise to an additional term in the electron Hamiltonian of the form:

$$(48) \quad H' = \frac{e}{mc} \sum_{\mathbf{n}} [(\mathbf{p}_{\mathbf{n}} \cdot \mathbf{A}) a_{\mathbf{n}+\boldsymbol{\lambda}_0}^* a_{\mathbf{n}} + (\mathbf{p}_{\mathbf{n}} \cdot \mathbf{A}^*) a_{\mathbf{n}-\boldsymbol{\lambda}_0}^* a_{\mathbf{n}}] +$$

$$+ \frac{e^2}{2mc^2} \sum_{\mathbf{n}} [\mathbf{A}^2 a_{\mathbf{n}+2\boldsymbol{\lambda}_0}^* a_{\mathbf{n}} + \mathbf{A}^{*2} a_{\mathbf{n}-2\boldsymbol{\lambda}_0}^* a_{\mathbf{n}}] + \frac{e^2}{mc^2} |\mathbf{A}|^2 \sum_{\mathbf{n}} a_{\mathbf{n}}^* a_{\mathbf{n}}.$$

If we gauge  $\mathbf{A}(\mathbf{x})$  to  $\text{div } \mathbf{A}(\mathbf{x}) = 0$ , i.e.  $A_x = 0$ , we can replace  $(\mathbf{p}_{\mathbf{n}} \cdot \mathbf{A})$  by

(\*) One can understand in a qualitative way the orders of magnitude of the two contributions to the free energy  $F_{c1}$ ,  $F_{c2}$  by considering the increase in energy due to a shifting of an outer layer of electrons on the Fermi sphere with thickness  $\propto (kT/\zeta)n_0$  by an amount of equal magnitude. The contribution to  $F_{c2}$  is then given by the portion of the shifted layer that lies outside the cylinder  $\mathbf{x}^2 = n_0^2$ ; as one easily sees, this gives rise to an additional factor  $(kT/\zeta)^{1/2}$  in  $F_{c2}$  over the total energy. (39) further shows that the average  $A_{\text{ext}}$  is of the order of the number of outside electrons, i.e.  $\propto (kT/\zeta)^{3/2}$ ; hence  $F_{c1}$  is  $\propto (kT/\zeta)^3$ .

<sup>(10)</sup> A. PAPAPETROU: *Phil. Mag.*, **42**, 95 (1951).



$(2\pi\hbar/L)(\mathbf{x} \cdot \mathbf{A})$ , and  $H'$  can be written in terms of boson operators:

$$(49) \quad H' = \frac{e^2}{mc^2} |\mathbf{A}|^2 \cdot N + \frac{e}{mc} \frac{2\pi\hbar}{L} \sum_{\mathbf{x}} [(\mathbf{A}\mathbf{x})\varrho_{\mathbf{x}\varepsilon_0+}^+ + (\mathbf{A}^*\mathbf{x})\varrho_{\mathbf{x},\varepsilon_0+}] + \\ + \frac{e^2}{2mc^2} \frac{2\pi\hbar}{L} \sum_{\mathbf{x}} [\mathbf{A}^2\varrho_{\mathbf{x},2\varepsilon_0+}^+ + \mathbf{A}^{*2}\varrho_{\mathbf{x},2\varepsilon_0+}],$$

(+ terms involving  $\mathbf{n}$  with  $\mathbf{x}^2 > n_0^2(1-\gamma)$ ). The total Hamiltonian (25) + (49) then has the following form:

$$(50) \quad H = (H_0 + H'_0) + (H_B + H'_B) + (H_c + H'_c)$$

where  $H_0$ ,  $H_B$ ,  $H_c$  are as in (38),

$$H'_0 = \frac{e^2}{mc^2} |\mathbf{A}|^2 \cdot N,$$

$$H'_B = \text{the terms involving } \varrho \text{ in (49),}$$

$$H'_c = \text{terms involving } \varrho_{\mathbf{n}} \quad (\mathbf{n}^2 > n_0^2(1-\gamma)).$$

No terms involving the  $\mathbf{A}_{\mathbf{x}\sigma}$  occur.

We are primarily only interested in  $H'_B$  and  $H'_0$ , these being the only terms which contribute at zero temperature.

By the canonical transformation

$$\varrho_{\varepsilon\mathbf{x}\sigma} \rightarrow \varrho'_{\varepsilon\mathbf{x}\sigma} = \varrho_{\varepsilon\mathbf{x}\sigma} + \frac{1}{\varphi(\mathbf{x})} \frac{e}{mc} \frac{2\pi\hbar}{L} (\mathbf{A}^* \cdot \mathbf{x}) \delta_{\varepsilon\varepsilon_0} + \frac{1}{\varphi(\mathbf{x})} \frac{e^2}{2mc^2} \frac{2\pi\hbar}{L} \mathbf{A}^{*2} \delta_{\varepsilon,2\varepsilon_0}, \\ (\text{and herm. conj.}),$$

$H_B + H'_B$  is transformed into

$$\sum_{\varepsilon\sigma\mathbf{x}} \varphi(\mathbf{x}) \varrho'_{\varepsilon\mathbf{x}\sigma} \varrho'_{\varepsilon\mathbf{x}\sigma} - \frac{e^2}{m^2c^2} \left( \frac{2\pi\hbar}{L} \right)^2 \sum_{\sigma\mathbf{x}} \frac{(\mathbf{A}^* \cdot \mathbf{x})(\mathbf{A} \cdot \mathbf{x})}{\varphi(\mathbf{x})} + \dots$$

The dots stand for terms of higher order in the magnetic field which do not interest us here. Evaluating the second term, again letting  $\gamma \rightarrow 0$ , we find that it is

$$= -N \frac{e^2}{mc^2} |\mathbf{A}|^2$$

thus exactly compensating  $H'_0$ . The Hamiltonian in presence of a magnetic field is therefore the same as the Hamiltonian (38) without field apart from

the term  $H'_c$  which affects only the «outer» electrons and is therefore only effective at higher temperatures, and apart from terms of higher order in the magnetic field strength.

Therefore our method of sound-waves does not yield the ordinary diamagnetism of the electron gas. This was to be expected, since diamagnetism (being an effect of the bulk of electrons) arises in ordinary calculations <sup>(3,10)</sup> from terms containing energy differences of the type

$$\frac{1}{2m} \left( \frac{2\pi\hbar}{L} \right)^2 [(n-\lambda)^2 - n^2 + (n+\lambda)^2 - n^2]$$

(with the above notations). Our approximation (18) however consistently neglects such effects.

*D) Density fluctuations.* — The problem to which BLOCH <sup>(7)</sup> first applied a sound-wave approach was the following: Consider the density

$$\varrho(\mathbf{x}) \equiv \psi^*(\mathbf{x})\psi(\mathbf{x}).$$

Its expansion in Fourier components can be written

$$(51) \quad \varrho(\mathbf{x}) = L^{-3} \sum_{\mathbf{v}} \varrho(\mathbf{v}) \exp \left( -i\mathbf{v}\mathbf{x} \cdot \frac{2\pi\hbar}{L} \right).$$

The problem of density fluctuations then reduces to the calculation of the expectation values

$$(52) \quad \langle \varrho^+(\mathbf{v})\varrho(\mathbf{v}') \rangle \equiv \Delta_{\mathbf{vv}'}$$

at a given temperature. We wish to show that these can readily be computed by our method for the case of  $|\mathbf{v}| \ll n_0$ : In fact, we note that for a  $\mathbf{v}$  parallel to the  $x$ -direction:

$$\mathbf{v} = (\varepsilon, 0, 0)$$

we have

$$\varrho(\mathbf{v}) = \sum_{\mathbf{x}} (\varrho_{\varepsilon\mathbf{x}^+} + \varrho_{\varepsilon\mathbf{x}^-}).$$

(This is not quite true, as on the right hand side the terms which contain electron transitions  $a_{n_1, n_2, n_3}^* a_{n_1 + \varepsilon, n_2, n_3}$  with  $n_1 < 0$ ,  $n_1 + \varepsilon > 0$  are missing; in our approximation I) II) III), however, these do not contribute).

Further we note that in (52) we can, by symmetry, let  $\mathbf{v}$  be parallel to  $x$ .

If then  $\mathbf{v}'$  is not parallel to  $x$ ,  $\varrho^+(\mathbf{v})\varrho(\mathbf{v}')$  induces a net change in the  $\Delta_{\mathbf{x}\sigma}$ -distribution, so that its diagonal elements and therefore its mean value vanish. We can therefore restrict ourselves to  $\mathbf{v}$  and  $\mathbf{v}'$  being both parallel to  $x$ . Then (52) becomes: (For brevity we restrict ourselves to zero temperature and let  $\gamma \rightarrow 0$ ; at higher temperatures the terms involving  $\mathcal{Q}_n$  contribute a small additional effect)

$$\begin{aligned}\Delta_{\mathbf{v}\mathbf{v}'} &= \langle (\sum_{\mathbf{x}} [\varrho_{\varepsilon\mathbf{x}+}^+ + \varrho_{\varepsilon\mathbf{x}-}])(\sum_{\mathbf{x}'} [\varrho_{\varepsilon\mathbf{x}'+} + \varrho_{\varepsilon\mathbf{x}'-}^+]) \rangle \\ &= \sum_{\mathbf{x}} \langle \varrho_{\varepsilon\mathbf{x}+}^+ \varrho_{\varepsilon\mathbf{x}+} + \varrho_{\varepsilon\mathbf{x}-} \varrho_{\varepsilon\mathbf{x}-}^+ \rangle.\end{aligned}$$

This readily gives

$$(53) \quad \Delta_{\mathbf{v}\mathbf{v}'} = \varepsilon \cdot \pi n_0^2 = \frac{3}{4} N \frac{\varepsilon}{n_0}$$

which is BLOCH's (7) result.

#### 4. - Conclusion.

The present method which is a straightforward generalization of TOMONAGA's (8) one-dimensional model, permits the construction of a « sound-wave » representation for any assembly of non-interacting fermions. The results are applicable to low-energy excitations only, which affect but the outermost layers of the Fermi distribution. Bulk effects (like e.g. diamagnetism), which depend upon the deviation of the electron energy from its linear approximation (18), cannot be treated by the present method.

The fact that the sound-wave excitations cannot fully represent the possible motions of the assembly, even at low energies, does not seem unnatural, as it might be argued that « random » motions will have to supplement the « collective » ones, as in the case of a gas with Coulomb repulsion (5). More serious is the asymmetry in the very definition of our sound-waves when applied e.g. to a momentum space representation. This implies that any isotropic interaction will necessarily induce changes in the distribution of the « random » variables. This is expected to hamper very seriously the treatment of interactions.

This asymmetry, on the other hand, seems to be necessary in order to avoid the difficulties connected with the possibility of « resonance-transitions » along the Fermi surface. Such transitions cannot be described in terms of sound-waves; they induce in general complicated changes in all the variables.

The author is greatly indebted to the «Swiss Arbeitsgemeinschaft für Stipendien in Mathematik und Physik» for a grant, to Professor H. FRÖHLICH for kind hospitality in his Department and for much valuable advice, and to Dr. J. M. BUCKINGHAM for interesting discussions.

#### RIASSUNTO (\*)

Descrizione di aggregati di fermioni non interagenti in termini di onde sonore. Per la completa determinazione del sistema occorrono dei gradi di libertà addizionali, ma si prevede che i loro contributi siano piccoli per ogni eccitazione isotropica. Vi si possono comprendere anche campi magnetici. Si calcolano le funzioni termodinamiche e si dà lo schema di una applicazione al problema delle fluttuazioni di densità.

(\*) *Traduzione a cura della Redazione.*

## Un elettromagnete di grande potenza per lastre nucleari.

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(ricevuto il 16 Novembre 1953)

**Riassunto.** — Viene descritto un elettromagnete studiato e realizzato allo scopo di produrre campi sufficientemente intensi da consentire misure dirette e sistematiche di deviazione magnetica sulle tracce delle particelle nelle emulsioni nucleari. Nelle esperienze preliminari compiute, con una potenza erogata di circa 800 kW il campo ottenuto in un interferro di 3 cm di larghezza e 10 cm di diametro ha superato i 50 000 gauss.

1. — Emulsioni sensibili nucleari, destinate alla rivelazione di particelle ionizzanti, sono state ultimamente da alcuni autori <sup>(1-6)</sup> introdotte in campi magnetici, nel tentativo di determinare il segno della carica ed anche la quantità di moto delle particelle stesse analogamente a quanto viene fatto colla camera di Wilson.

Le condizioni sperimentali vengono ad essere però ben diverse da quelle realizzate nelle esperienze con la camera di Wilson in campo magnetico. In

(<sup>1</sup>) I. BARBOUR: *Phys. Rev.*, **76**, 320 (1949).

(<sup>2</sup>) C. FRANZINETTI: *Phil. Mag.*, **41**, 86 (1950).

(<sup>3</sup>) Y. GOLDSCHMIDT-CLERMONT and M. MERLIN: *Nuovo Cimento*, **7**, 220 (1950).

(<sup>4</sup>) M. MERLIN, B. VITALE and Y. GOLDSCHMIDT-CLERMONT: *Nuovo Cimento*, **9**, 421 (1952).

(<sup>5</sup>) C. C. DILWORTH, S. J. GOLDSACK, Y. GOLDSCHMIDT-CLERMONT and F. LEVY: *Phil. Mag.*, **7**, 1032 (1950).

(<sup>6</sup>) C. C. DILWORTH and S. J. GOLDSACK: *Nuovo Cimento*, **7**, 926 (1953).



primo luogo la lunghezza media dei percorsi delle particelle nelle emulsioni è piccola, dai 2 ai 5 mm, mentre nella camera di Wilson varia dai 10 ai 30 cm; in secondo luogo la densità del mezzo,  $4 \text{ g/cm}^3$ , è tale che la deviazione subita dalla particella negli urti con gli atomi dell'emulsione, non è affatto trascurabile rispetto a quella dovuta all'azione del campo magnetico, mentre lo è nel gas della camera di Wilson con densità dell'ordine di  $10^{-3} \text{ g/cm}^3$ . Ne risulta ovviamente che per ottenere deviazioni suscettibili di dare le informazioni richieste, sono necessari campi magnetici estremamente elevati, in volumi che, se appaiono piccoli rispetto a quelli richiesti dalla camera di Wilson, non lo sono tuttavia in senso assoluto.

I più potenti elettromagneti finora progettati ed attuati per la camera di Wilson producono campi magnetici dai 5 ai 10 000 gauss, in volumi variabili dai 10 ai 40 dm<sup>3</sup>, con potenze che vanno dai 10 ai 100 kW. Con tali elettromagneti, nel volume richiesto dalle esperienze con lastre nucleari, si può giungere al più sui 30 000 gauss; ragione per cui gli sperimentatori hanno finora tentato vari artifici per superare le sopradette difficoltà.

Allo scopo di ridurre la densità del mezzo alcuni autori <sup>(1-4)</sup> hanno usato le emulsioni nucleari con intercapedini di qualche millimetro d'aria a pressione normale o ridotta. La curvatura delle traiettorie ha luogo così in un mezzo di densità trascurabile; essa viene determinata mediante le tangenti alle tracce lasciate dalle particelle nelle due emulsioni affacciate. Altri autori <sup>(5,6)</sup> invece hanno tentato di usare opportune emulsioni a densità più bassa.

Allo scopo di aumentare la lunghezza delle tracce a disposizione, si sono usate emulsioni spesse fino ad 1,2 mm. Attualmente è in istudio l'uso di pile di emulsioni libere (senza supporto di vetro) che opportunamente ricomposte permetteranno di portare all'ordine dei centimetri i tragitti utili delle particelle.

Il fattore predominante resta comunque l'aumento del campo magnetico. Il proporzionale incremento nella curvatura delle tracce porta ad aumento notevole sia nel numero delle particelle misurabili, che in quello degli eventi nucleari totalmente analizzabili: la probabilità di incontrare  $n$  tracce associate suscettibili di un esame soddisfacente cresce infatti come  $H^{2n}$  <sup>(6)</sup>.

Ci siamo proposti perciò di realizzare un elettromagnete, specialmente studiato per emulsioni sensibili, tale da poter fornire, con la potenza di circa 800 kW a disposizione, una induzione magnetica sull'ordine dei 60 000 gauss in servizio continuativo ed in un volume di 240 cm<sup>3</sup> (10 cm di diametro e 3 cm di interferro).

Per dare all'apparecchio, in previsione di eventuali altre possibili applicazioni, un più largo impiego è stata prevista sin d'ora la possibilità di un aumento di potenza, in particolare per uso intermittente, e, dal punto di vista meccanico, è stata prevista la possibilità di variazioni nelle dimensioni, distanze e forma delle testate terminali.

2. - La realizzazione di un elettromagnete, come quella di qualsiasi macchina, deve essere affrontata in ordine al problema economico nei suoi due contrastanti aspetti, del costo di costruzione e di quello di esercizio. L'economia di costruzione porta necessariamente ad un maggior costo di esercizio, per la più elevata potenza assorbita dal circuito di eccitazione. Quando poi si debba ricorrere ad una conversione di energia, entra in considerazione anche il costo del macchinario di conversione.

Per giudicare della bontà o meno dei criteri costruttivi e per giustificare la scelta è dunque necessario premettere le condizioni specifiche del problema. Nel caso in questione risultava disponibile, senza spesa, il macchinario di conversione della corrente alternata in continua per una potenza massima di 1000 kW ed era prevista una utilizzazione molto limitata nel tempo, cioè non più di un mese all'anno, cosicché il costo dell'energia dissipata, ove avesse dovuto essere computato, risultava comunque limitato. Per contro scarsi erano i mezzi disponibili per la costruzione dell'apparecchio. Risulta da ciò la giustificazione di una costruzione impostata col criterio di minimo costo compatibile con i limiti della potenza disponibile, sopra accennati.

Stabilito, come dato di partenza, il volume utile d'interferro, si è cercato, per successive approssimazioni, di assegnare al ferro ed alle bobine la disposizione più opportuna, tenendo conto di un altro dato di fatto e cioè che la potenza elettrica veniva messa a disposizione alla tensione di 200-220 V con una corrente di  $4\,500 \div 5\,000$  A.

È il caso di affermare subito che risultati assai più soddisfacenti si sarebbero ottenuti se, fissa restando la potenza disponibile, fosse stato possibile scegliere liberamente il valore della corrente, che vantaggiosamente avrebbe potuto essere dell'ordine di 10 000 e più ampere. Comunque si è cercato subito di stabilire il valore massimo della densità di corrente impiegabile, ricorrendo ad un avvolgimento in rame tubolare a sezione esterna quadrata con foro di sezione circolare per il passaggio dell'acqua.

Sulla base di prove preliminari si è controllata la validità delle ordinarie formule dell'idraulica per tubi di sezione assai piccola (diametro da 5 a 8 mm) con velocità dell'acqua molto elevata.

Con queste esperienze, sulle quali non ci intratteniamo, è stata accertata la possibilità di adottare, in ottime condizioni di raffreddamento, densità di corrente dell'ordine di 60 e più A/mm<sup>2</sup>.

Stabilito questo, siamo passati alla scelta della disposizione generale dell'avvolgimento, fino ad arrivare per successive approssimazioni a quella definitiva. È ovvio che le migliori condizioni si ottengono rendendo minima la lunghezza dei nuclei magnetici e rendendo minimo il diametro delle spire.

Queste due condizioni sono fra loro in contrasto per cui non è facile rendersi immediatamente conto della soluzione più vantaggiosa. Infatti la localizzazione dell'avvolgimento su un diametro limitato porta di necessità ad

un allungamento dei nuclei e quindi ad un aumento delle amperspire richieste; una distribuzione di esso su una lunghezza assiale limitata porta ad un mag-

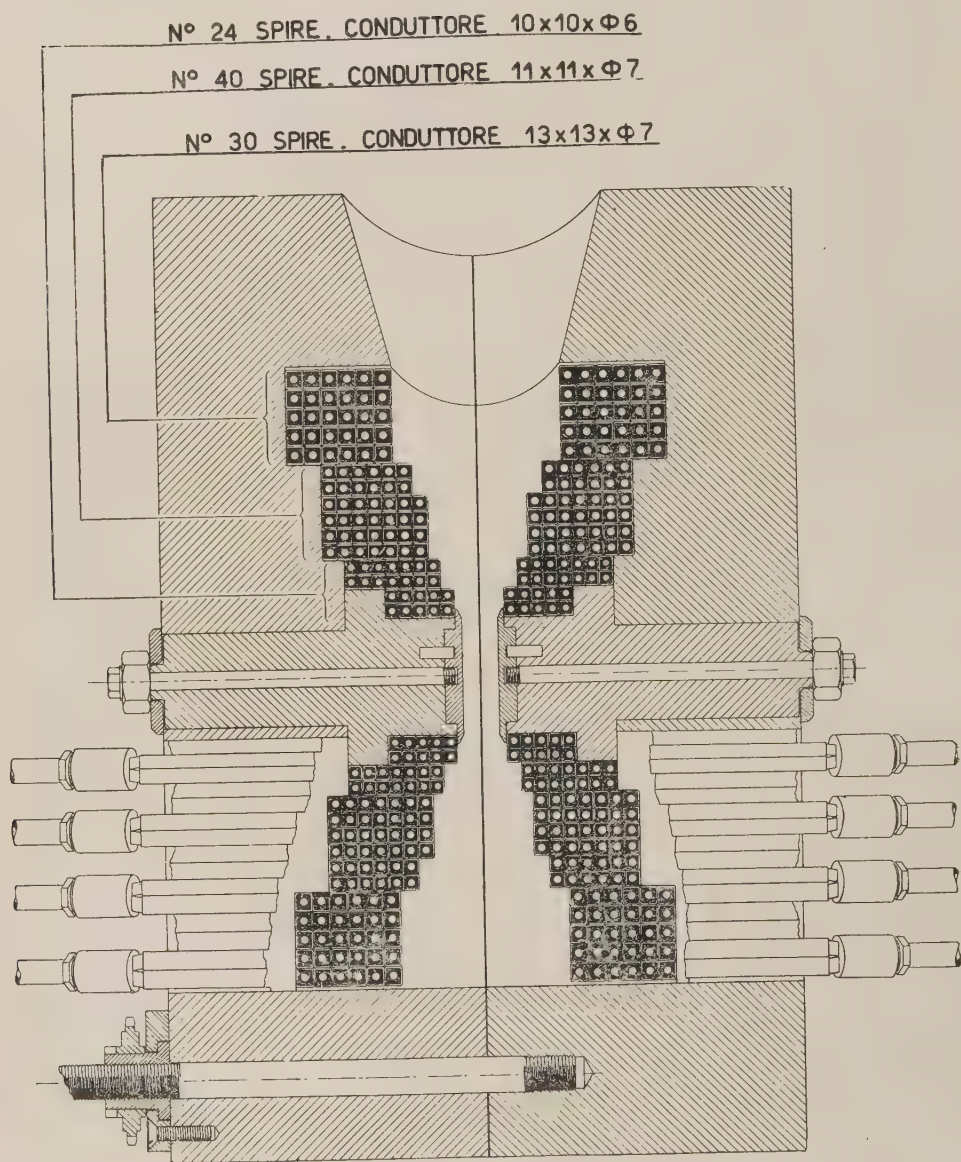


Fig. 1.

giore diametro medio delle spire e quindi ad una maggiore dissipazione di energia e ad un più elevato flusso disperso, con conseguente aumento del peso di ferro.



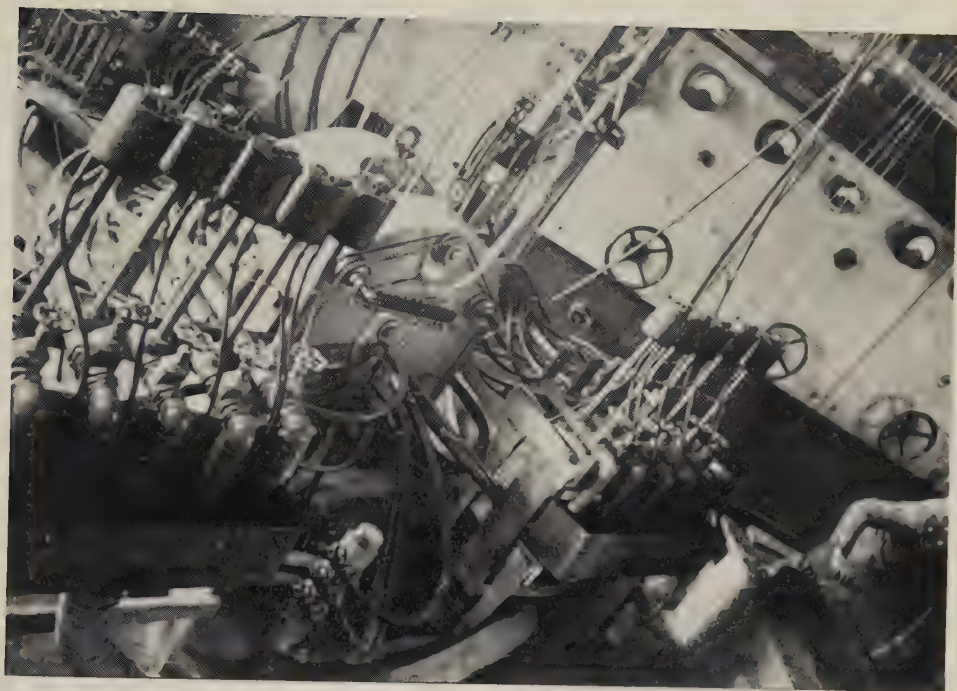


Fig. 2.

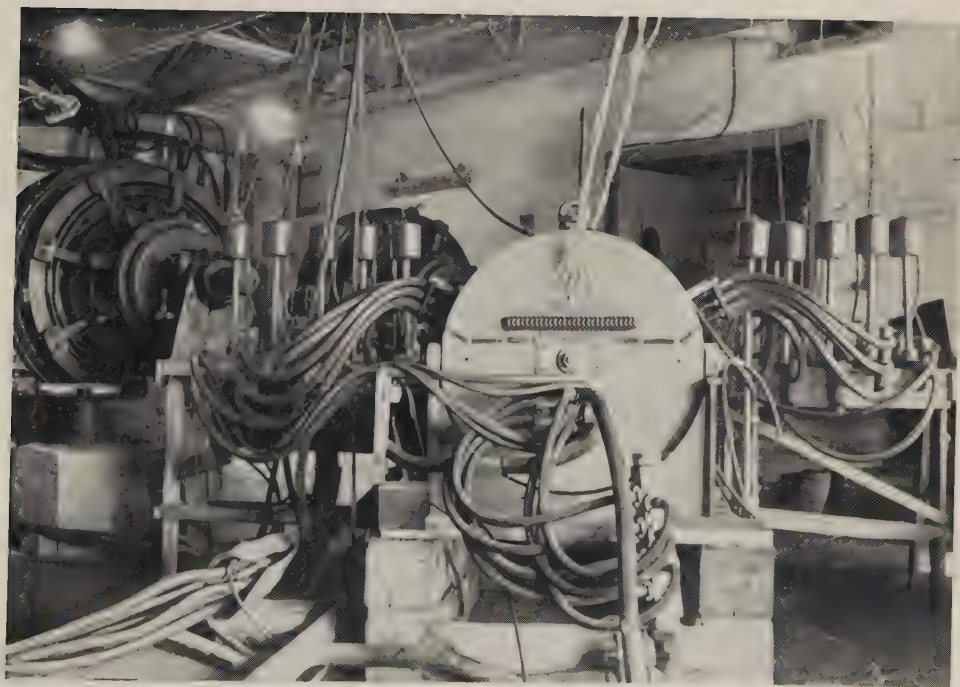


Fig. 3.





La brevità di questa nota ci impedisce di illustrare minutamente i criteri di controllo e di calcolo seguiti nel predisporre numerosi piani costruttivi, vincolati ovviamente dalla possibilità di esecuzione pratica dell'avvolgimento.

Si sono studiate soluzioni con diverse forme di avvolgimento, considerando anche l'impiego di conduttori a sezione costante o variabile.

Ci limitiamo a constatare che non sono emerse, fra le varie soluzioni, grandi differenze. Quella definitivamente adottata è chiaramente illustrata dalla fig. 1 nella quale l'elettromagnete è riprodotto in sezione longitudinale. Esso consta di due corpi cilindrici in acciaio fuso del diametro esterno di 820 mm che realizzano un circuito magnetico a ferro chiuso, simmetrico intorno all'asse delle espansioni polari.

Una asimmetria, magneticamente di scarso rilievo, è determinata dall'apertura nella parte superiore della carcassa ( $30^\circ$  nel piano assiale e  $60^\circ$  nel piano ortogonale all'asse) destinata a consentire il libero ingresso alla radiazione proveniente dall'angolo solido così definito. Le due metà della carcassa sono normalmente chiuse a contatto per effetto della stessa attrazione magnetica, ma ovviamente sono anche collegate con vitoni distribuiti lungo la periferia e comandati simultaneamente da una catena, visibili nella fotografia di fig. 2; essi hanno anche la funzione di provocare il distacco delle due parti, quando ciò si renda necessario per operazioni di manutenzione, e si debba vincere l'attrazione dovuta a magnetizzazione residua. Metà della carcassa è fissa e l'altra metà scorre a mezzo di piccole ruote a gola su due rotaie solidali col basamento visibili nella fotografia di fig. 3.

I nuclei polari, in acciaio dolce fucinato, sono indipendenti dalla carcassa ed eventualmente sostituibili con altro materiale.

L'avvolgimento è costituito di  $2 \times 94$  spire, cosicchè l'eccitazione risulta  $188 \times 5000 = 940\,000$  amperspire, riferita alla corrente nominale di 5000 A.

Tutte le spire sono elettricamente collegate in serie, mentre le necessità di raffreddamento hanno imposto la suddivisione del circuito in  $2 \times 14 = 28$  vie d'acqua in parallelo. Gli scarichi d'acqua (non visibili nel disegno) avvengono nella parte inferiore interna di ciascun gruppo di bobine.

Si sono impiegati conduttori tubolari di tre diverse sezioni, come indicato nella fig. 1, e ciò allo scopo di equilibrare la dissipazione di energia nelle singole sezioni dell'avvolgimento onde ottenere una portata d'acqua pressochè eguale, a pari perdita di carico, in tutti i 28 circuiti di raffreddamento in parallelo. L'adduzione della corrente è stata realizzata, a partire dalle sbarre rigide di alimentazione, mediante un conduttore flessibile cordato di sezione assai limitata ( $120\text{ mm}^2$ ) posto entro un tubo di gomma e raffreddato da energica circolazione d'acqua; la stessa acqua raffredda anche i morsetti di ingresso e di uscita, e nello stesso modo è realizzato il collegamento in serie delle due bobine onde consentire la rapida separazione delle due metà della struttura, come sopra indicato.

Il peso totale dell'elettromagnete è di kg 2135 così ripartito:

— Acciaio magneticamente attivo	kg. 1 800
— Supporti in ferro	» 200
— Rame dell'avvolgimento	» 135

Particolare importanza è stata data alla sicurezza della circolazione d'acqua, in quanto un arresto anche di pochi secondi porterebbe alla produzione di

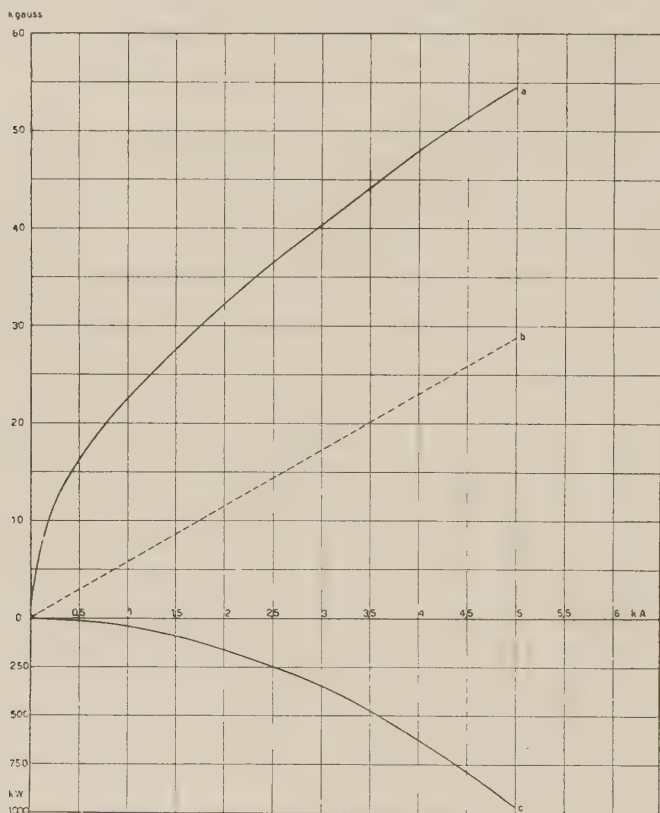


Fig. 4. — curva *a*: induzione nell'interfero; curva *b*: campo in aria; curva *c*: potenza erogata.

vapore e al danneggiamento degli avvolgimenti, specie in corrispondenza alle saldature dei diversi elementi. Sono stati installati 10 indicatori di circolazione d'acqua (1 per ogni 3 circuiti) del tipo a galleggiante, muniti di contatto elettrico.

Una riduzione di portata, in uno qualsiasi dei circuiti d'acqua, determina l'immediata apertura dell'interruttore generale.

In corrispondenza a ciascun tratto del circuito, formante una sezione idraulicamente indipendente, sono stati derivati conduttori di piccola sezione, ripor-

tati a due morsettiere (visibili nella fotografia di fig. 3) e di qui al banco di controllo, dal quale viene periodicamente letta la caduta di tensione in ciascuna sezione, mediante un voltmetro commutabile su ognuna di esse. Questo controllo è stato fatto per rivelare l'eventuale formazione di incrostazioni attraverso l'aumento di resistenza dovuto ad accrescimento della temperatura del conduttore.

3. — Le prove di funzionamento hanno nel loro complesso fornito risultati assai soddisfacenti. Il problema che più preoccupava a priori era naturalmente quello della refrigerazione. E si può dire che sotto questo riguardo i risultati

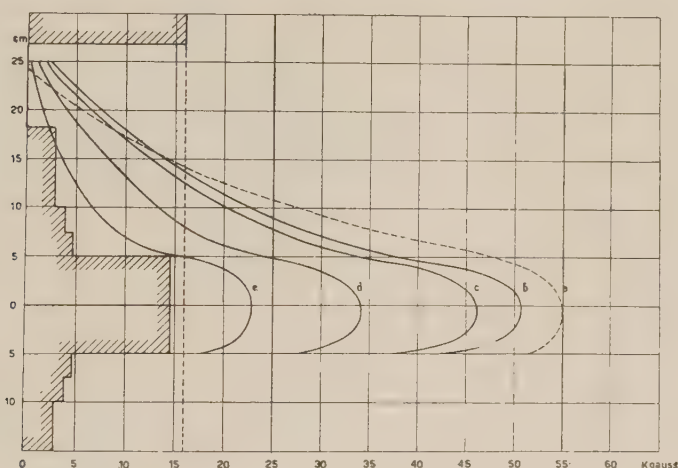


Fig. 5. — curva *a*: calcolata per 5000 A; curva *b*: misurata per 4470 A; curva *c*: misurata per 3200 A; curva *d*: misurata per 2270 A; curva *e*: misurata per 1060 A.

sono stati particolarmente favorevoli. L'acqua, prelevata dall'acquedotto a 4 °C con una portata di circa 8 l/s entrava nel sistema dei 28 circuiti in parallelo ad una pressione di 12 atmosfere. La temperatura d'uscita s'aggravava sui 24 °C, con scarti di qualche °C fra i diversi circuiti.

I risultati sono compendati nei seguenti diagrammi. Nella fig. 4 sono riportati in funzione dell'intensità di corrente i valori massimi dell'induzione nell'interferro e i valori della potenza erogata. Il massimo valore dell'induzione raggiunta è stato di 52300 gauss con una corrente di 4750 A e una potenza di 825 kW. La curva tratteggiata pure riportata sulla fig. 4 rappresenta l'andamento calcolato del campo in aria.

Nella fig. 5 sono riportati i valori dell'induzione ottenuta con varie correnti di eccitazione lungo il diametro verticale passante per il centro dell'interferro. Nella fig. 6 è invece tracciato l'andamento delle linee di induzione costante

sul piano di simmetria normale all'asse per una corrente di 4460 A. Il cerchio tratteggiato indica il contorno dell'espansione polare. L'insieme di questi due diagrammi permette la completa conoscenza dell'inomogeneità del campo. In

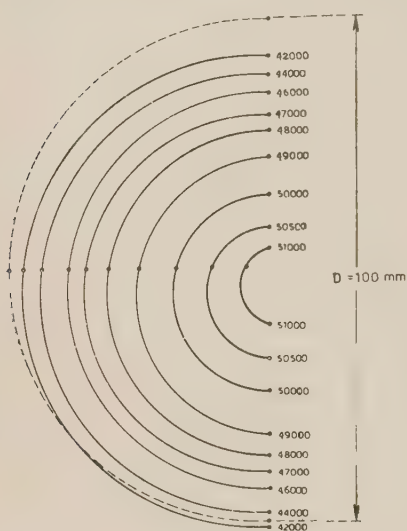


Fig. 6.

genere si può dire che il risultato sotto tale punto di vista è molto soddisfacente, le massime variazioni del campo sino a pochi mm dal contorno dell'espansione polare, e quindi praticamente sull'intera superficie utile delle lastre nucleari, essendo contenute entro il 10%. Si può notare che vi è una certa asimmetria tra l'alto ed il basso, dovuta all'apertura superiore della carcassa: il centro del sistema di linee a  $B$  costante appare leggermente spostato verso il basso rispetto al centro dell'espansione. Per correnti di eccitazione inferiori, l'inomogeneità risulta nettamente diminuita, come si può vedere dalla fig. 5. In ogni modo, la conoscenza delle linee di uguale induzione permette di riportarne l'andamento sulle lastre stesse affacciate all'espansione, e quindi di attribuire a

queste punto per punto il valore esatto del campo. La rapida sostituzione delle attuali testate terminali dei nuclei polari con testate di profilo opportunamente studiato permetterà di ridurre considerevolmente l'attuale inomogeneità del campo, se eventuali future esperienze lo richiederanno.

L'elettromagnete ha funzionato complessivamente, nel corso di queste prove, per 40 giorni effettivi, con brevi intervalli d'interruzione, dovuti a cause secondarie. Si può dire che non vi sono stati inconvenienti di rilievo. Pacchi di lastre sono stati esposti a due riprese nel campo di oltre 50 000 gauss ottenuto. L'esplorazione di queste, attualmente in corso, ha confermato le previsioni circa l'utilità del metodo.

4. — La realizzazione di questo elettromagnete è stata possibile grazie all'aiuto generoso della SADE e delle Officine Galileo che ne fecero dono all'Istituto di Fisica, della Montecatini che mise a disposizione i trasformatori e le convertitrici per l'alimentazione, e dell'Azienda elettrica municipale di Milano che ospitò l'intero impianto presso i suoi stabilimenti di Digapoli offrendo l'energia necessaria ed ogni sorta di facilitazioni utili al successo dell'esperienza. Ringraziamenti particolari vadano agli Ingg. M. MAINARDIS, direttore centrale e V. ANTONELLO, segretario generale della SADE, all'ing. A. ALOCCO, direttore dello Stabilimento di Battaglia Terme delle Officine Galileo, all'ing. A.

LUMACHI, progettista di questo, per il valido aiuto prestato alla risoluzione dei problemi tecnologici, al dott. L. MORANDI, vicepresidente della Montecatini, all'ing. C. CASTELLANI, direttore dei servizi elettrici, all'ing. F. CARATI, direttore generale dell'A.E.M., ed ai suoi collaboratori ing. C. CHIERICHETTI e sig. B. TRINCA. Ci è infine grato ringraziare il prof. A. ROSTAGNI, direttore dell'Istituto di Fisica, per il costante interessamento ed appoggio col quale ha seguito le varie fasi dell'esecuzione e funzionamento della complessa apparecchiatura.

#### S U M M A R Y

An electromagnet has been studied and realized with the aim of producing sufficiently high fields to permit direct and systematic measurements of magnetic deflection of particles in nuclear emulsions. For a power consumption of 800 kW, a field of 50 000 gauss has been attained in the disc-shaped volume between the pole pieces 3 cm in depth and 10 cm in diameter.



## On the Nature of the $\kappa$ -Meson.

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(ricevuto il 2 Dicembre 1953)

**Summary.** — Assuming that the  $\kappa$ -meson is a fermion which interacts with other fermions through the universal Fermi interaction, the decay probability and the capture probability by nuclei are calculated. The obtained values are in rather good agreement with the experimental data till now available.

Recent experimental evidence obtained with nuclear plates has confirmed the existence of one or more heavy mesons with a mass of about  $1000 m_e$ , which disintegrate according to different decay schemes. Actually the mass measurements are not yet sufficiently accurate to decide whether it is always the same particle which decays in different manners or whether there are different particles with approximately the same mass. Owing to this uncertainty and to the lack of precise data, it seems rather difficult at the present time to ascertain the real nature of the neutral decay products in some types of decays as the  $\kappa$  or  $\chi$  type. If we adopt the first hypothesis that the  $\kappa$  and  $\chi$  decays are due to the same particle which decays also according to the  $\tau$  scheme <sup>(1)</sup>, it becomes necessary to postulate the following equivalent reactions:

$$(1) \quad \left\{ \begin{array}{l} \tau \rightarrow 3\pi \\ \chi \rightarrow \pi + \gamma \\ \kappa = \mu + \nu_{\mu} + \gamma. \end{array} \right.$$

<sup>(1)</sup> C. F. POWELL: *Report of the Congress international sur le Rayonnement cosmique*, Bagnères de Bigorre, p. 221, 1953; B. ROSSI: The same *Report*, p. 259; M. GOLDHABER: to be published on *Phys. Rev.*

If we adopt the second hypothesis instead, it is not necessary to suppose that the particle which decays according to the  $\kappa$  type is a boson and it can then be assumed that the decay follows the scheme:

$$(2) \quad \kappa \rightarrow \mu + \nu + \nu.$$

In this case, the  $\kappa$ -meson would obviously be a fermion; and the photon, which in some experimental cases appears to be emitted in a direction opposed to that of the charged product of the decay <sup>(2)</sup> should be attributed exclusively to the decay of a  $\chi$ .

We do not want here to take any definite position regarding these two hypotheses we propose only to examine some of the consequences which would follow directly from the second one, i.e. from the assumption that the  $\kappa$ -meson decays according to reaction (2).

The experimental evidence seems to indicate that the interaction between any pair of fermions is always governed by the same constant of interaction <sup>(3)</sup>  $g = \sim 10^{-19} \text{ erg} \cdot \text{cm}^3$ . If we assume that this same constant acts also in the interaction of the  $\kappa$ -particle with any other fermion, we can easily calculate its mean life for decay and capture by nuclei and compare these values with those relative to the decay and capture of the  $\mu$ -meson.

We may calculate the reciprocal of the mean life  $\lambda$ , according to the formula:

$$\lambda = \frac{2\pi}{\hbar} |H'|^2 \varrho(E),$$

in which  $|H'|$  is the matrix element of interaction and  $\varrho(E)$  is the density of the final states per unit energy interval, indicating with  $\mu$  the mass of the decaying particle, with  $m$  the mass of the charged decay product and assuming a zero mass for the neutrons. We obtain for the matrix element:

$$H' \sim \frac{g}{\Omega} \quad (\Omega \text{ volume of the space considered})$$

and for the density of final states with momentum  $P$  and kinetic energy  $T$  for the charged decay particle:

$$\varrho(E) = \frac{1}{2\pi^2} \frac{P^2}{\hbar^3} \Omega \frac{dN}{dW},$$

where  $\frac{dN}{dW} = \frac{\Omega}{48} \frac{cm^2}{\pi^2 \hbar^3} [3(w-1)^2 - 2(3w-2)T + 2T^2], \quad \text{and} \quad w = \frac{\mu}{m}.$

<sup>(2)</sup> B. ROSSI: *Report of the Congrès international sur le Rayonnement cosmique*, Bagnerès de Bigorre, p. 107, 1953.

<sup>(3)</sup> G. PUPPI: *Nuovo Cimento*, 6, 194 (1949).

Integrating over all values of  $P$  one obtains:

$$\lambda = \frac{g^2 c^4 \mu^5}{384 \pi^3 \hbar^7} F(w),$$

$$\text{where } F(w) = \frac{1}{4} \left( 1 - \frac{1}{w^4} \right) \left( 1 - \frac{8}{w^2} + \frac{1}{w^4} \right) + \frac{6 \log w}{w^4} + \frac{1}{10} \left( 1 - \frac{1}{w^2} \right)^5.$$

This gives, for the relative rate of decay of the  $\kappa$  and  $\mu$ -mesons:

$$(3) \quad \frac{\lambda_{\kappa d}}{\lambda_{\mu d}} = \frac{g_{\kappa}^2}{g_{\mu}^2} \frac{\mu_{\kappa}^5}{\mu_{\mu}^5} \frac{F(w)_{\kappa}}{F(w)_{\mu}}.$$

Assuming the values

$$g_{\kappa} = g_{\mu}, \quad \tau_{\mu} = 2.15 \cdot 10^{-6} \text{ s}, \quad w_{\kappa} = 5, \quad w_{\mu} = 210,$$

one obtains for the mean life of the  $\kappa$ -meson:

$$(4) \quad \lambda_{\kappa d} = 2380 \lambda_{\mu d},$$

hence

$$\tau_{\kappa d} = 9 \cdot 10^{-10} \text{ s}.$$

A same result has been recently obtained also by DAVIDSON <sup>(4)</sup>.

The mean life of the  $\kappa$ -meson is not yet known with accuracy; the actual data from plate and cloud chamber work seem <sup>(5)</sup> to indicate a value of about  $10^{-9}$  s in quite good agreement with the value calculated above.

We shall now assume that the capture of a negative  $\kappa$  meson at the end of its range by a nucleus follows the same mechanism as the capture of a  $\mu$ -meson. After having reached the  $K$  orbit of the atom, the  $\kappa$ -meson interacts with a proton of the nucleus according to the reaction:

$$\kappa^- + P = N + \nu.$$

Indicating with  $M$  the mass of the nucleon and with  $\mu$  that of the meson, we have also in this case for the matrix element:

$$H' \sim \frac{g}{\Omega}.$$

<sup>(4)</sup> J. P. DAVIDSON: *Phys. Rev.*, **91**, 1020 (1953).

<sup>(5)</sup> W. L. ALFERD and R. B. LEIGHTON: *Phys. Rev.*, **90**, 622 (1953).

while the density of final states turns out simply to be:

$$\varrho(E) = \frac{\Omega P_n^2}{2\pi^2 \hbar^3 (c + v_n)} = \frac{\Omega}{2\pi^2 \hbar^3} \mu^2 c f(u)$$

$$\text{where } f(u) = \frac{\left(1 + \frac{u}{2}\right)^2}{(1 + u)^2 u \left(1 + \frac{u}{2}\right) + \sqrt{(1 + u)^2 + u^2 \left(1 + \frac{u}{2}\right)^2}},$$

$v_n$  and  $P_n$  are the velocity and the momentum of the neutron and  $u = \mu/M$ . We obtain then, for the probability of interaction with the  $Z$  protons:

$$\lambda = \frac{g^2 P_n^2 Z}{\pi \hbar^4 \Omega (c + v_n)}.$$

The volume  $\Omega$  may be written as:

$$(5) \quad \frac{Z}{\Omega} = n \int_0^R |\psi|^2 dv.$$

With  $R$  the nuclear radius,  $n$  the density of the nucleons and  $\psi$  the eigenfunction of the meson on the  $K$  orbit.

With the same symbols previously used, we obtain for the ratio of the probabilities of capture for the  $\kappa$  and  $\mu$ -mesons:

$$(6) \quad \frac{\lambda_{\kappa c}}{\lambda_{\mu c}} = \frac{g_{\kappa}^2 \mu_{\kappa}^2 f(u_{\kappa}) \int_0^R (\psi_{\kappa})^2 dv}{g_{\mu}^2 \mu_{\mu}^2 f(u_{\mu}) \int_0^R (\psi_{\mu})^2 dv}.$$

We have not however taken into account the effect of the Pauli principle which forbids capture processes with too little momentum transfer to the nucleon; the cut-off factor will be certainly greater in the case of the  $\kappa$  than in the case of the  $\mu$  and therefore the ratio has to be considered as a lower limit which has to be multiplied by a factor presumably not greater than 2.

According to BONETTI and TOMASINI<sup>(6)</sup>, the proportion of  $\mu^-$  captures which occur in heavy nuclei is about 50%. We shall assume the same proportion for the  $\kappa$  captures. We shall also assume the positive excess to be zero for the low energies of the mesons considered here, and shall take the

(6) A. BONETTI and G. TOMASINI: *Nuovo Cimento*, **8**, 693 (1951).

data relative to silver as characteristic for the heavy nucleus capture, and those of oxygen for the light nucleus capture.

The integral  $\int_0^R |\psi_\mu|^2 dv$  for the  $\mu$  case has been calculated by WHEELER <sup>(7)</sup> for various elements and may be expressed by using the effective atomic number  $Z_{\text{eff}}$ . The capture probability for the  $\mu$ -meson may then be expressed by the relation

$$(7) \quad \lambda_{\mu c} = \lambda_{\mu d} \left( \frac{Z_{\text{eff}}}{Z_0} \right)^4.$$

where  $Z_0 = 11$ . Deviations from this law due to shell structure <sup>(8)</sup> of the nucleus would be, according to experimental data, quite unimportant in this case.

For the  $\kappa$ -meson, in the case of Ag, the  $K$  orbit is completely inside the nucleus and it will be a fairly good approximation to take for its eigenfunction the one associated with the ground state of the harmonic oscillator. We obtain then:

$$\int_0^R |\psi_\kappa|^2 dv = \frac{2}{\sqrt{\pi}} [\varphi(\xi) - \xi e^{-\xi^2}] \quad \text{where} \quad \varphi(\xi) = \int_0^\xi e^{-\xi^2} d\xi \quad \xi = \sqrt{\frac{e}{2\hbar}} (Z_\mu R)^{1/4}.$$

For oxygen, the  $K$  orbit is outside the nuclear radius; it will be sufficient for obtaining the order of magnitude to take for integral (5) a mean value between the values obtained with a harmonic oscillator and a hydrogen-like wave function.

With  $g_\kappa = g_\mu$   $u_\kappa = 0,575$ ,  $u_\mu = 0,115$  we find in this way:

$$(8) \quad \begin{cases} \lambda_{\mu c}^{Ag} = 38 \lambda_{\mu d}, & \lambda_{\kappa c}^{Ag} = \sim 1220 \lambda_{\mu d} \\ \lambda_{\mu c}^0 = 0,22 \lambda_{\mu d} & \lambda_{\kappa c}^0 = \sim 330 \lambda_{\mu d} \end{cases}$$

We see from formulas (3) (6) and (8) that, while the decay probability increases with the 5th power of the mass of the decaying fermion, the capture probability increases only more or less faster than the 2nd power according to atomic number. In any case decay will be comparatively more probable for the heavier mesons than capture. According to formulas (4) and (8) the proportions of the total number of  $\kappa$  and  $\mu$ -mesons which decay  $D$  and are

<sup>(7)</sup> J. A. WHEELER: *Rev. Mod. Phys.*, **21**, 133 (1949).

<sup>(8)</sup> J. W. KEUFFEL, F. B. HARRISON, T. N. GODFREY and G. T. REYNOLDS: *Phys. Rev.*, **87**, 942 (1952); J. M. KENNEDY: *Phys. Rev.*, **87**, 953 (1952).



captured  $C$  turn out to be:

$$\begin{aligned} D_{\mu} &= 0,7 & C_{\mu} &= 0,3 \\ D_{\kappa} &= 0,89 & C_{\kappa} &= 0,11 . \end{aligned}$$

The proportion of the decaying  $\kappa$  is very high, and this could therefore explain the fact that until now very few heavy meson capture stars have been found; some of which should also be attributed to the capture of the negative  $\tau$ -mesons. As for the negative  $\kappa$  captured, according to the preceding hypothesis, they should give rise to rather small disintegration stars, for of the 500 MeV energy released in the capture of the  $\kappa$ , 370 would go to the neutrino, and only 130 to the neutron. This type of star should appear rather similar to the normal  $\sigma$  stars due to  $\pi$ -meson capture, and probably not easily distinguishable from them.

#### RIASSUNTO

Nell'ipotesi che la particella  $\kappa$  sia un fermione, si calcolano la costante di disintegrazione spontanea e la probabilità di cattura da parte di un nucleo, assumendo che la particella  $\kappa$  interagisca cogli altri fermioni tramite l'interazione universale di Fermi. I valori ottenuti si prestano ad interpretare in modo soddisfacente i dati sperimentali finora esistenti.

# LETTERE ALLA REDAZIONE

(La responsabilità scientifica degli scritti inseriti in questa rubrica è completamente lasciata dalla Direzione del periodico ai singoli autori)

## Zur renormierten einzeitigen Bethe-Salpeter-Gleichung.

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(ricevuto il 16 Novembre 1953)

Zur näherungsweisen Lösung der mehrzeitigen Bethe-Salpether-Gleichung <sup>(1)</sup> (B.S.Gl.) kann es günstig sein, sie in eine einzeitige Integralgleichung umzuformen. LÉVY <sup>(2)</sup> und KLEIN <sup>(3)</sup> führen dies durch Einsetzen der «adiabatischen Näherung» in die hinreichend oft iterierte B.S.Gl. aus; MACKE <sup>(4)</sup> zieht vor jedem neuen Iterationsschritt einen Teil ab, der in der Graphenschreibweise einem einzeitig reduzierbaren Graphen entspricht (siehe unten). Das letzte Verfahren ist der Lösung einer inhomogenen Integralgleichung durch die Neumannsche Reihe verwandt und daher möglicherweise verbesserungsfähig.

Eine renormierte einzeitige Gleichung erhält man am einfachsten, wenn man die erwähnte Umformung an der bereits renormierten mehrzeitigen B.S.Gl. ausführt <sup>(5)</sup>. Deren Renormierung geschieht nämlich genau wie die der *S*-Matrix <sup>(6)</sup> und besteht in der Vermeidung aller Selbstenergie- und Scheitelteile sowie dem Gebrauch ausschließlich der Funktionen  $S'_F(x-x')$ ,  $\Delta'_F(x-x')$  und  $\Gamma(x, x', y)$  an Stelle der Funktionen  $S_F(x-x')$ ,  $\Delta_F(x-x')$  und  $\gamma\delta(x-y)\delta(x'-y)$ . Jene Funktionen sind (wenigstens in der Quantenelektrodynamik) als prinzipiell bekannt anzusehen und von KARPLUS und KROLL <sup>(6)</sup> bis zur zweiten Näherung schon berechnet worden. Im Anschluß an MACKE <sup>(4)</sup> und ZIMMERMANN <sup>(7)</sup> ist im folgenden die Herleitung der einzeitigen Gleichung kurz angedeutet, wobei eine wohl ohne weiteres verständliche Graphenschreibweise (z.B. ZIMMERMANN <sup>(7)</sup>) benutzt werden soll.



Fig. 1,

Fig. 1 zeigt die mehrzeitige renormierte B.S.Gl. Einfache gerichtete Linien stellen  $S'_F$ -Funktionen dar, der umrandete Teil ist die aus  $S'_F$ ,  $\Delta'_F$ , und  $\Gamma$  zusammengesetzte Funktion  $\bar{G}$  von vier Argumenten von B. und S. und wird dargestellt durch die Gesamtheit von im Sinne von B.

<sup>(1)</sup> E. E. SALPETER and H. A. BETHE: *Phys. Rev.*, **84**, 1232 (1951).

<sup>(2)</sup> M. M. LÉVY: *Phys. Rev.*, **88**, 72 (1952).

<sup>(3)</sup> A. KLEIN: *Phys. Rev.*, **90**, 1101 (1953).

<sup>(4)</sup> W. MACKE: *Phys. Rev.*, **91**, 195 (1953); *Nuovo Cimento*, **10**, 1198 (1953).

<sup>(5)</sup> F. J. DYSON: *Phys. Rev.*, **75**, 1736 (1949).

<sup>(6)</sup> R. KARPLUS and N. M. KROLL: *Phys. Rev.*, **77**, 536 (1950).

<sup>(7)</sup> W. ZIMMERMANN: *Nuovo Cimento*, im Erscheinen.

und S. irreduziblen und renormierten Feynman-Graphen. An die Stelle der unteren Pfeilspitzen ist die links des Gleichheitszeichens stehende Funktion zweier Argumente wieder einzusetzen. Iteration und Gleichsetzen der Zeiten gibt Fig. 2. Hierbei bedeuten waagrechte gestrichelte Linien gleiche Zeit für alle darauf liegenden Eckpunkte. Sonst sind alle Zeitreihenfolgen willkürlich bis auf die durch Schlangenlinie bezeichneten (spätere Zeit oben).

Nun wird

$$S'_F(x'_2 - x''_2) = \int S_F(x'_2 - z) \{ \delta(z - x''_2) + \Sigma_C^* S'_F(z - x''_2) \} dz$$

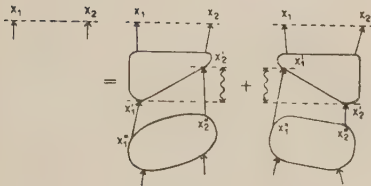


Fig. 2.

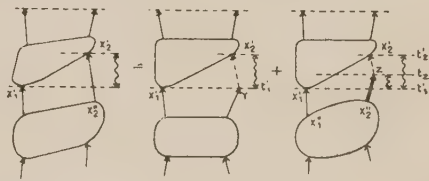


Fig. 3.

bei  $x'_2$  (bzw.  $x'_1$ ) eingesetzt. Dabei ist  $\Sigma_C^*$  der « konvergente eigentliche Selbstenergie-  
teil » von DYSON (5). Die über den ganzen Raum zu erstreckende Integration über  $z$  wird aufgeteilt nach  $t_z > t'_1$  und  $t_z < t'_1$ . Im zweiten Fall läßt sich zufolge

$$S_F(x'_2 - z) = \int S_F(x'_2 - y) \beta dy S_F(y - z) \quad (t_{x'_2} \geq t_y \geq t_z)$$

die Integration über  $x''_2$  in eine solche über  $dy$  ( $t_y = t'_1$ ) umwandeln, siehe Fig. 3. Im ersten Graphen der rechten Seite kann nämlich nach Fig. 1 die einzeitige Wellenfunktion eingesetzt werden. Die gestrichelte Linie stellt die Funktion  $S_F$ , die Doppellinie die Funktion

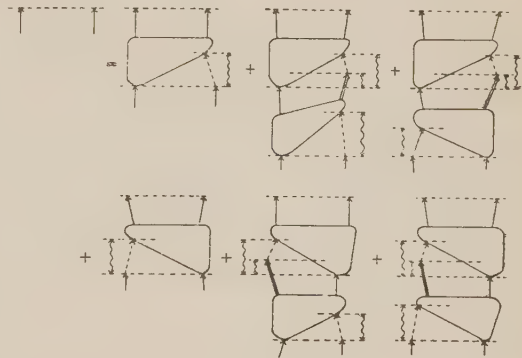


Fig. 4.

$$(i\gamma_\nu \partial_\nu - m) S'_F(z - x''_2) = \delta(z - x''_2) + \Sigma_C^* S'_F(z - x''_2)$$

dar. Durch weiteres Iterieren erhält man schließlich (unter Vernachlässigung des Restgliedes) die einzeitige Integralgleichung der Fig. 4.

Die Zerlegungsvorschrift für einen beliebigen Graphen des einzeitigen B.S. Typs (d.h. Ein- und Auslauf wie bei den Graphen der Fig. 4) ist in folgender Definition (7) enthalten, die aus Fig. 4 unmittelbar abgelesen werden kann: An einer Stelle mehrzeitiger Reduzibilität im Sinne von B. und S. besteht auch einzeitige Reduzibilität, wenn die folgenden Bedingungen a) und b) erfüllt sind: Durchschneidet man die

Fermionlinie am früheren Fermionen-Einlaufpunkt des darüberliegenden mehrzeitig irreduziblen Graphen und bei der gleichen Zeit an einer geeigneten Stelle die andere Fermionenlinie, so muß *a*) der Graph in zwei nicht mehr zusammenhängende Teilgraphen zerfallen und *b*) der obere Teilgraph in der zweiten durchschnittenen Fermionenlinie frei von Selbstenergieteilen sein, wobei für die  $S_F'$ -Funktion hier die oben gegebene Darstellung einzusetzen ist. Wenn es eine solche Stelle gibt, ist sie

durch diese Bedingungen eindeutig bestimmt und die zweite Fermionenlinie dort aufsteigend. Die rechte Seite der Fig. 4 erfaßt genau alle im obigen Sinne einzeitig irreduziblen Graphen.

Bisher ist die volle 16-komponentige Wellenfunktion verwendet worden. Das Schema für die positiven Frequenzanteile allein zeigt Fig. 5. Die Fortsetzung geschieht ebenso wie früher. Die Zerlegungsvorschrift lautet jetzt wie oben bis auf den Zusatz, daß an dem zeitlich früheren Fermionen-Einlaufpunkt ein positiver Frequenzanteil ( $S_F'$ -Funktion von früherer Zeit her) liegen muß.

Ein anderer ebenso konsequenter Weg zur einzeitigen Gleichung besteht in der Verwendung von  $S_R$ - an Stelle von  $S_F$ -Funktionen in Fig. 3, 4, 5. Die Zeitreihenfolgen bleiben die gleichen, jedoch brauchen sie jetzt zum Teil wegen der Eigenschaft der  $S_R$ -Funktion nicht mehr explizit berücksichtigt zu werden, wie aus den Graphen leicht zu entnehmen. Im zweiten, dritten, fünften und sechsten Graphen auf der rechten Seite von Fig. 4 z.B. ist nur noch eine Zeitreihenfolge zu beachten.

Nochmals sei hervorgehoben, daß bei den Graphen alle Punkte, soweit nicht ausdrücklich eine Aussage über die Zeitreihenfolge gemacht ist, über das ganze Raum-Zeit-Gebiet zu integrieren sind, was für diese einfache Durchführung der Renormierung entscheidend ist. Nur an Einlaufteil und Zwischenstücken der Graphen treten zeitliche Integrationsbegrenzungen auf, jedoch die wenigstens in der Leiternäherung nicht zu einer Divergenz des Integralkerns führen.

Eine größere Ähnlichkeit mit der Graphengesamtheit der Famm-Dancoff-Methode <sup>(2,7,8)</sup> erhält man, wenn man die B.S.Gl. wie folgt umformt (*a*, *b* bezeichnen die beiden Fermionen, Matrixmultiplikation wird vorausgesetzt):

$$\begin{aligned} \varphi &= S_F'^{(a)} S_F'^{(b)} \bar{G} \varphi = \\ &= (1 - S_F \Sigma^{*(a)})^{-1} (1 - S_F \Sigma^{*(b)})^{-1} S_F^{(a)} S_F^{(b)} \bar{G} \varphi \\ &= \{ S_F \Sigma^{*(a)} + S_F \Sigma^{*(b)} - (S_F \Sigma^{*(a)})(S_F \Sigma^{*(b)}) + S_F^{(a)} S_F^{(b)} \bar{G} \} \varphi. \end{aligned}$$

<sup>(9)</sup> M. CINI: *Nuovo Cimento*, **10**, 526 u. 614 (1953).

Wendet man hier das oben geschilderte Verfahren an, so erhält man Selbstenergie-teile sowohl für sich allein als auch an der Einlaufseite der Graphen (so z.B. bei MACKE <sup>(4)</sup>). Das letztere kommt jedoch auf eine Entwicklung der  $S'_F$ -Funktion nach Potenzen der Kopplungskonstanten hinaus, was ungünstig sein kann <sup>(9)</sup>. Daher scheint die so erhaltene Form der B.S.Gl. gegenüber der ursprünglichen keine Vorteile zu bieten.

(<sup>9</sup>) K. A. BRUECKNER, M. GELL-MANN und M. GOLDBERGER: *Phys. Rev.*, **90**, 476 (1953).



## On the Production of Electronic Component by Fast $\mu$ -Mesons.

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(ricevuto il 19 Novembre 1953)

In previous communications <sup>(1,2)</sup> we have reported some results on the production of penetrating particles by  $\mu$ -mesons, obtained in a cloud chamber experiment underground. In the present paper we give data referring to the production of electronic component by  $\mu$ -mesons.

This work has been performed with the photographs obtained during the first part <sup>(1)</sup> of the previous research. The chamber, of useful dimensions  $40 \times 50 \times 15$  cm<sup>3</sup>, containing 9 lead plates each 1.6 cm thick, was located under  $\sim 55$  m w.e. of granite. Counter control consisted in a simple telescope above the chamber triggered by a single ionizing particle able to penetrate 8 cm of lead at least.

One can assume that the penetrating particles incident on the chamber are  $\mu$ -mesons with energies distributed according to a differential spectrum  $N(E) dE = 2E_0^2(E + E_0)^{-3} dE$  <sup>(3)</sup>, where  $E_0$  is the ionization loss across the rock. In

our case

$$N(E) dE = 338(E + 13)^{-3} dE,$$

the average energy being about 15 GeV <sup>(4)</sup>.

In the field of the low energies we have valued the rate of production of delta-rays in the gas of the chamber (argon at 90 cm Hg and 20 °C). We have counted 150 delta-rays ranging at least 3 mm, i.e. electrons of energy greater than about 20 keV, in a total observed path length of 52 g/cm<sup>2</sup> of argon (810 photographs examined). The experimental value of 3 delta-rays per g/cm<sup>2</sup> of argon is in satisfactory agreement with the value 3.4 from the Rutherford formula <sup>(5)</sup>.

In the field of the intermediate energies we have studied the rate of production of knock-on electrons emerging from the lead plates, their angular distribution and the production of successive knock-ons.

(i) *Rate of production of knock-on electrons in lead.* — We examined 1500 photographs taking into account only singly emerging electrons, which do not scatter remarkably in the gas of the chamber, i.e. of energy greater than

<sup>(1)</sup> A. LOVATI, A. MURA, C. SUCCI and G. TAGLIAFERRI: *Nuovo Cimento*, **10**, 105 (1953).

<sup>(2)</sup> A. LOVATI, A. MURA, C. SUCCI and G. TAGLIAFERRI: *Nuovo Cimento*, **10**, 1201 (1953) and *Rep. of Bagnères de Bigorre Conference* (July 1953), in course of publication.

<sup>(3)</sup> E. P. GEORGE and J. EVANS: *Proc. Phys. Soc.*, A **63**, 1248 (1950).

<sup>(4)</sup> In <sup>(1)</sup> the value of the average energy of  $\mu$ -mesons was wrongly reported as 40 GeV.

<sup>(5)</sup> B. ROSSI: *High Energy Particles* (New York, 1952).

about 1 MeV. Taking into account the effect of the plate thickness (see for example <sup>(6)</sup>) and the influence of the energy of the incident mesons on the cross-section for collision processes and on the spectrum of knock-on electrons <sup>(5)</sup>, our results are in agreement with those of the other authors (see Table I).

form  $\cos^n \theta$ , with  $n=2.5$ , fits well the experimental data. The same value for  $n$  was obtained by BROWN *et al.* <sup>(7)</sup> in different experimental conditions. The agreement seems to show that the angular distribution is determined mainly by the scattering properties of the material of the plates.

TABLE I. — Rate of production of knock-on electrons in lead by  $\mu$ -mesons, obtained with cloud chamber experiments.

Authors	Average energy of incident mesons (Gev)	Thickness of lead plates (cm)	Number of transversals	Rate of singly emerging knock-on electrons	
				forwards %	backwards %
BROWN <i>et al.</i> <sup>(7)</sup> . . .	2 (*)	1.27	3364	6	—
			3304	—	1.8
WALKER <sup>(8)</sup>	7	1.27	3519	9	—
WOLFENDALE <sup>(9)</sup> . . . (under rock)	10	1	8748	5.2	—
LOVATI, MURA and SUCCI (under rock)	15	1.6	11980	6.0	.5
TIFFANY and HAZEN <sup>(10)</sup> (under rock)	100	2.54	776	21	—
			808	—	1

(\*) Estimated by the present writers.

(ii) *Angular distribution of knock-on electrons emerging forwards from the lead plates.* — The projected angle  $\theta$  between the direction of the incident meson and the direction of the singly emerging secondary electron was measured in 500 cases. An empirical relationship of the

(iii) *Production of successive knock-on electrons in lead.* — AMALDI *et al.* <sup>(11)</sup> with a counter hodoscope have observed an unexpectedly high rate of production of « double electronic secondaries » (d.e.s.) from  $\mu$ -mesons.

We have counted the pictures showing one penetrating particle producing in different plates of the chamber either 2 or 3 or 4 single knock-ons in the forward direction. The values obtained are in very good agreement with the ones to be expected from a statistically independent production of knock-on elec-

<sup>(6)</sup> P. BASSI and A. LORIA: *Nature*, **163**, 400 (1949).

<sup>(7)</sup> W. W. BROWN, A. S. MCKAY and E. D. PALMATIER: *Phys. Rev.*, **76**, 506 (1949).

<sup>(8)</sup> W. D. WALKER: *Phys. Rev.*, **90**, 234 (1953).

<sup>(9)</sup> A. W. WOLFENDALE: private communication.

<sup>(10)</sup> O. L. TIFFANY and W. E. HAZEN: *Phys. Rev.*, **77**, 849 (1950).

<sup>(11)</sup> E. AMALDI, C. CASTAGNOLI, A. GIGLI and S. SCIUTI: *Nuovo Cimento*, **9**, 969 (1952).

trons (Table II). We have moreover considered both single knock-ons and electronic showers produced by  $\mu$ -mesons

TABLE II. — *Production of successive single knock-on electrons in forward direction.*

Successive knock-ons	Number of pictures	
	values to be expected (*)	observed values
2	101.2	97
3	13.9	17
4	1.2	2

(\*) The distribution of the 1500 pictures examined, according to the number of plates crossed in each picture, is as follows:  $N_4 = 49$ ,  $N_5 = 89$ ,  $N_6 = 111$ ,  $N_7 = 150$ ,  $N_8 = 286$  and  $N_9 = 815$ .

(total rate per traversal of lead: 7.1%), but once more no significant difference

events result indistinguishable, but it is easy to detect them in the case of an energy transfer greater than 1 GeV. In fact an electron (or a photon or a pair of electrons) of the energy of 1 GeV produces a cascade showing 8-10 electrons at the maximum and developing through three plates at least.

We have examined 2500 photographs taking into account only the six upper plates of the chamber: in 14860 traversals we observed 13 electronic showers of estimated energy greater than 1 GeV. The rate per  $\text{g/cm}^2$  of lead results

$$k_{\text{exper}} = (.5 \pm .2) \cdot 10^{-4},$$

corresponding to a cross-section  $\sigma_{\text{exper}} = (8 \pm 3) \cdot 10^{-29} \text{ cm}^2/\text{nucleon}$ . The statistics are rather poor, nevertheless it must be emphasized that the detection of the mesons interacting in the plates is

TABLE III. — *Probability of energy transfer greater than 1 GeV from  $\mu$ -mesons by electronic component production (per  $\text{g/cm}^2$  of lead).*

Authors	Average energy of incident mesons (GeV)	Number of trasversals	Thickness of lead plates (cm)	Probability	
				$k_{\text{theor}} \cdot 10^4$	$k_{\text{exper}} \cdot 10^4$
LOVATI, MURA and SUCCI	15	14860	1.6	.42	.5 $\pm$ .2
HAZEN <sup>(13)</sup> . . . . .	100	4143	2.5	2.1	3.0 $\pm$ .7

between observed and expected values was obtained. Analogous result was obtained recently by WOLFENDALE <sup>(12)</sup>.

In the field of the high energies the production of electronic component from  $\mu$ -mesons is known to be due to collision, radiation and pair production processes. In our experimental conditions these

not favoured and that the identification of the showers is ensured by the good geometrical conditions and by the chosen thickness of the plates.

The experimental value can be compared with the theoretical one calculated from the energy distribution of the  $\mu$ -mesons and from the cross-sections of collision, radiation and pair production processes. For the integrated probability of energy transfer greater than 1 GeV we have obtained

$$k_{\text{theor}} = .42 \cdot 10^{-4}.$$

<sup>(12)</sup> A. W. WOLFENDALE: *Nuovo Cimento* **10**, 1493 (1953). We thank dr. WOLFENDALE for having communicated us his paper before publication.

<sup>(13)</sup> W. E. HAZEN: *Phys. Rev.*, **86**, 764 (1952).

The results of HAZEN for  $\mu$ -mesons of average energy about 100 GeV and ours are shown in Table III.

The comparison between theoretical and experimental values is indicative that, if an additional mechanism leading to the production of electronic component exists, it should not be very important. If this is so, the main phenomenon of the interaction with the Coulomb field of the nucleus is accounted for by the present knowledge about  $\mu$ -mesons: it seems interesting to remember that also the production of pe-

netrating particles <sup>(14)</sup> — due to the interaction of  $\mu$ -mesons with the mesonic field — is consistent with the theoretical predictions <sup>(15)</sup>.

<sup>(14)</sup> E. P. GEORGE and J. EVANS: loc. cit.; E. AMALDI *et al.*: loc. cit.; A. LOVATI *et al.*: loc. cit.; M. DEUTSCHMANN: *Zeits. f. Naturfor.*, **8a**, 308 (1953). The last results of V. A. APPAPILLAY, A. W. MAILGANAM and A. W. WOLFENDALE, as communicated by dr. WOLFENDALE at the Cagliari Conference (September 1953), also agree with the results of the above mentioned authors.

<sup>(15)</sup> S. HAYAKAWA: *Phys. Rev.*, **84**, 37 (1951).

## La vita media del mesone $\tau$ .

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(ricevuto il 25 Novembre 1953)

Dai dati finora raccolti sul decadimento  $\tau$  ( $\tau^\pm \rightarrow \pi^\mp + 2\pi^\pm$ ) risulta una vita media dell'ordine di almeno  $10^{-9}$  s. È noto che in questo caso, come negli altri casi di particelle che decadono per interazione col campo mesonico, la vita media risulta molto più lunga ( $\sim 10^{12}$  volte) di quanto potrebbe prevedere la teoria dei campi; tuttavia è stato proposto di interpretare tale discordanza con l'intervento di uno spin elevato della particella instabile. La spiegazione è analoga a quella di WEIZSÄCKER per gli isomeri nucleari: uno spin elevato ha per conseguenza un alto momento angolare orbitale delle particelle uscenti, il che riduce notevolmente la funzione d'onda a piccola distanza relativa.

Ciò è vero se si suppone che le funzioni d'onda delle particelle prodotte possano essere approssimate con

$$\frac{(pr)^L}{(2L+1)!!},$$

dove  $p$  è il momento,  $L$  il momento angolare,  $r$  la distanza relativa (\*); questo è lecito se le dimensioni della regione dove l'interazione è forte (presumibilmente dell'ordine del raggio Compton

del  $\tau$ ) sono abbastanza piccole, il che accade in questo caso ( $pr \lesssim 1/3$ ). Si può inoltre supporre che l'elemento di matrice del decadimento sia proporzionale al valore medio, fatto con un peso imprecisato, della funzione d'onda nella regione suddetta. I calcoli dettagliati mostrano che basta porre  $L = 4$  per spiegare l'ordine di grandezza della vita media sperimentale.

Tenuto presente quanto precede, è di un certo interesse il fatto che si può ottenere un comportamento soddisfacente riguardo alla vita media anche per un mesone di spin 1 e parità dispari: ciò è dovuto alle regole di selezione sulla parità, come ora mostreremo.

Il decadimento del  $\tau$  avviene in due particelle identiche ed una di carica opposta. Le prime due, essendo bosoni dovranno trovarsi in uno stato di parità spaziale  $+$ , e quindi di momento angolare relativo  $l$  pari. La parità totale del sistema delle due particelle identiche è perciò sempre positiva. La terza particella abbia, rispetto al baricentro delle prime due, momento angolare  $\lambda$ . La parità totale delle tre particelle è allora  $-(-)^{\lambda}$  (i  $\pi$  sono pseudoscalari). Pertanto, se il  $\tau$  ha parità  $+$ ,  $\lambda$  deve essere dispari e viceversa. Se la parità del  $\tau$  è  $-$ ,  $l$  e  $\lambda$  sono entrambi pari; se poi lo spin  $J$  è dispari né  $l$  né  $\lambda$  possono essere zero,

(\*) Definizioni più precise di queste grandezze saranno date in un lavoro più esteso sull'argomento.



dovendo in tal caso essere o  $J=l$  o  $J=\lambda$ ; pertanto il minimo valore di  $l$  e di  $\lambda$  è 2. Si vede dunque che anche per  $J=1$ , se la parità del  $\tau$  è  $-$ , l'andamento delle funzioni d'onda dei  $\pi$  per piccole distanze è quella stessa che si avrebbe per  $L=4$ .

Lo spettro dei  $\pi$  emessi nel decadimento del  $\tau$  è abbastanza sensibile al momento angolare del  $\tau$  <sup>(1)</sup>; pertanto l'esame dei dati sperimentali dovrebbe permettere di distinguere almeno tra i valori  $J=0$  e  $J=1$ .

È opportuno osservare a questo punto che le regole di selezione usate (parità e momento angolare) non escludono il decadimento  $\tau^\pm \rightarrow \pi^\pm + \pi^0$  per  $J=1$  e parità  $-$  <sup>(1)</sup>. Tale processo può però essere escluso, assumendo l'ipotesi della «charge independence»; nel modo seguente. Supposto possibile sia il decadimento in tre  $\pi$  che quello in due, risulta altresì possibile la reazione:  $\pi^+ + \pi^0 \rightarrow \pi^+ + \pi^+ + \pi^-$  attraverso lo stato intermedio  $\tau$ . Ma tale reazione è proibita da una regola di selezione debole (v. PAIS e JOST <sup>(2)</sup> oppure MICHEL <sup>(3)</sup>) (intendiamo per regole di selezione de-

boli quelle che non valgono più quando si ammette l'intervento dell'interazione elettromagnetica). Notiamo che il ragionamento precedente è basato sull'ipotesi che il  $\pi^0$  abbia spin isotopico 1 (e quindi accoppiamento  $\tau_3$ ), mentre non fa alcuna ipotesi sulla struttura del  $\tau$ . Se il  $\tau$  fosse una particella vettoriale con accoppiamento vettoriale o tensoriale al campo nucleonico, risulterebbe permesso il decadimento in due  $\pi$  e proibito quello in tre; tuttavia occorre notare che una particella di spin 1 e parità  $-$  non deve essere necessariamente vettoriale, e che il suo accoppiamento col campo nucleonico può non essere vettoriale nè tensoriale. Queste considerazioni mostrano l'opportunità di considerare il  $\tau$  piuttosto come uno stato del campo dei mesoni e dei nucleoni, che come una particella elementare; questo modo di vedere è del resto coerente con le idee attuali sui mesoni pesanti.

Tutta la questione verrà riesaminata più completamente e con maggior dettaglio in un lavoro in preparazione; osserviamo solo, per concludere, che la discussione precedente lascia ancora insoluta la questione del decadimento del  $\tau$  in due  $\pi$  ed un fotone, e di tutti gli altri processi di natura elettromagnetica, che non sono proibiti dalle regole di selezione e potrebbero ridurre notevolmente la vita media del  $\tau$ .

<sup>(1)</sup> R. H. DALITZ: *Phil. Mag.*, **44**, 1068 (1953).

<sup>(2)</sup> A. PAIS e R. JOST: *Phys. Rev.*, **87**, 871 (1952).

<sup>(3)</sup> L. MICHEL: *Nuovo Cimento*, **10**, 319 (1953).

## An Analysis of Cosmic Ray Jets.

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(ricevuto il 30 Novembre 1953)

In a recent paper <sup>(1)</sup> DILWORTH *et al.* gave the results of their recent experimental work on cosmic ray jets and, inter alia, compared their results with the theory of high energy cosmic ray jets due to ourselves <sup>(2)</sup>. There are a number of features of this paper which, we feel, require comment.

In our paper we pointed out that at sufficiently high energies a nucleon may be expected to punch a hole through a nucleus which it strikes. For energies above 16  $A$  (in GeV,  $A$  is the atomic weight) the tunnel will be nearly cylindrical. For lower energies the tunnel (if this concept still applies cf. HEITLER and TERREAUX <sup>(3)</sup>) will flare out. We dealt quantitatively only with the first (high energy) case and it is only to this case that our formulae can apply. We gave  $\gamma_p \geq 500$  as an estimate of the limit for application to events in the photographic plate.

In their comparison DILWORTH *et al.*

use the data of 68 events. Of these only a dozen or so have  $\gamma_p \geq 500$  GeV. We wish to stress that the others of lower energy cannot be interpreted by means of our formulae which assume the cylindrical tunnel. The few high energy events seem to be in good agreement with our theory. For the low energy events only crude qualitative predictions can be made from our model. Since the tunnel is then trumpet-shaped a greater part of the target nucleus will be involved and thus more nucleons struck than in the case of the cylindrical tunnel. This explains the higher multiplicities for some of the lower energy jets found both by DILWORTH *et al.* and previously by the Bristol group <sup>(4)</sup>. An increase in the average multiplicity with *decreasing* energy in this region has also been found by WALKER <sup>(5)</sup> using carbon in a cloud chamber.

It is perhaps worth noting, that the exact behaviour of the average multiplicity for a central collision depends on whether one assumes a theory of multiple or plural production of mesons in the individual nucleon-nucleon encounters.

(\*) Now with I.C.I. Ltd., Welwyn, England.

<sup>(1)</sup> C. C. DILWORTH, S. J. GOLDSACK, T. F. HOANG and L. SCARSI: *Nuovo Cimento*, **10**, 1261 (1953).

<sup>(2)</sup> F. C. ROESLER and C. B. A. McCUSKER: *Nuovo Cimento*, **10**, 127 (1953).

<sup>(3)</sup> W. HEITLER and C. TERREAUX: *Proc. Phys. Soc.*, A **66**, 929.

<sup>(4)</sup> R. R. DANIEL, J. DAVIES, J. H. MULVEY and D. H. PERKINS: *Phil. Mag.*, **43**, 453 (1952).

<sup>(5)</sup> W. D. WALKER: private communication.

This is particularly noticeable for energies above  $16 A$  i.e. for cases in which the tunnel is cylindrical. Here a plural theory predicts a constant average multiplicity whilst a multiple theory predicts an average multiplicity increasing with energy. The dependence on the atomic weight of the target is also different on the two assumptions, such that «plural» theories are likely to entail a faster increase of  $n_s$  with  $A$  than «multiple» theories. The two cases have been dealt with by ourselves <sup>(6)</sup> and by HEITLER and TERREAUX <sup>(3)</sup>.

(\*) C. B. A. McCUSKER and F. C. ROESLER:  
*Phys. Rev.*, **91**, 769 (1953).

Finally, it does not follow from our theory that «the maximum tunnel length can be determined from the median angle of the jet» and more specially it can not be given by equation (13) of DILWORTH *et al.*

Apart from fluctuations, the maximum tunnel length is given by the diameter of the struck nucleus. The minimum is given by  $s = s_1$ . It is obvious that eqn. (13) of DILWORTH *et al.* can give tunnel lengths of arbitrarily many nucleon diameters for sufficiently large  $\gamma_p$  and also tunnel lengths much less than 1 nucleon diameter for sufficiently small  $\gamma_p$ .

## A Punched Card System of Recording of Events in Nuclear Emulsion Work.

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(ricevuto il 2 Dicembre 1953)

During the scanning of cosmic ray plates, a considerable amount of information is accumulated in numerous scanning books on many types of events. In order to select special events for measurement or to make statistics, it is convenient to have events recorded in some way by which the selection can be automatic. When events are recorded on cards, the sorting of these cards by hand is very tedious.

We have therefore made a preliminary trial of a system of punched cards, which carry much of the physical description of the event in the form of perforations, enabling the events to be selected according to various parameters <sup>(1)</sup>. For simplicity we have chosen a system of cards with a double row of holes round the edges, in which events are selected by passing knitting needles through the holes. Using a special clipper a certain number of these holes (depending on the type of event) is cut away to the edge. To select events of a chosen type

the needles are passed horizontally through a stack of cards, through the relevant holes. On lifting the needles the cards which have been cut at these holes fall away, while the others remain hanging on the needles. The principal information carried on the card is repeated in writing on the centre to avoid errors in cutting passing unnoticed.

In the system we have adopted the events are classified in 4 categories: 1) stars; 2) electromagnetic events; 3) fundamental particles; 4) events associated with the preceding events.

For most of the sections we have adopted a triangular enumeration <sup>(2)</sup>. A number is indicated by two cuts. It lies in the diamond determined by the intersection of the two diagonal rows ending at the cuts. The upper digit is indicated when the cut on the right includes both holes, the lower one if the left cut is long.

To indicate the number of branches in a star, we have a different system of enumeration. The selection most frequently required is that  $N_s$  should be greater than a certain number and  $N_b$

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<sup>(1)</sup> We are indebted to Dr. A. BONETTI of the University of Milan for certain suggestions which we have adopted in the design of the card.

<sup>(2)</sup> COX, CASEY and BAILEY: *Journ. of Chem. Educ.*, **24**, 65 (1947).

less than a certain number. A differential method such as used for the other sections would require a large number of sortings to draw all the stars

The system can be most simply explained by an example. Fig. 1 represents a card which describes a star of twelve branches of which four are shower particles ( $N_s = 4$ ), the eight others being grey or black tracks ( $N_h = 8$ ). One of the black branches is a hammer track.  $N_s$  has been cut away up to and including 4 (Integral classification) and  $N_h$  from 20 to 8 included. The existence of a hammer track is indicated under « associated events »; this code contains 20 spaces, the code number of the hammer track being 1.

This section is represented by a single row of holes, so that any number of associated phenomena can be represented simultaneously.

Another information on stars is contained in the section « primary » in which is indicated the presence of a visible primary and its nature when determined (integral numeration). In our example, there is no definite primary and the section has not been cut.

The hole « is secondary » is cut when the event on the card is secondary to an event registered on another card.

The fundamental particles and the electromagnetic events are classified by the triangle system in compartments of 20 and 10 places respectively, part of which are empty to leave room for new types of events.

The hole « measurable tracks » is cut when one or more tracks are measurable. The hole « measured » is cut when some of the tracks have actually been measured. In the example described there were 4 measurable tracks but no measured tracks.

The plate number (309) is given again on a decimal classification using three triangles with room for 1000 plates.

The classification of cosmic ray events in nuclear emulsions on punched cards seems to us useful and the cards described adequate to carry the necessary

The diagram shows a punched card with various sections and data fields. At the top, there is a row of 20 holes labeled 1 through 20. Below this, the card is divided into several sections:

- Associated Events** (top section, 20 holes)
- Observer:**
  - Von den Camp
  - Plate No. 309
  - Book: 1
  - Page: 53
- Type of event:**
  - Star
  - $N_s = 4$   $N_h = 8$
  - Hammer track
- Code:**
  - Associated
  - E (visible)
- Measurable tracks:** 4
- Total length measured by:**
- Is secondary of:**
- In plate No.:**
- Associated Events:**
  - N
  - In plate No.
- Fund. Part. E.M. Events:**
  - Fund.
  - Part.
  - E.M. Events
- measurable tracks measured:**
- Associated Events** (bottom section, 20 holes)

The card also features a triangle system for classification and a diagram of a star with 12 branches.

Fig. 1.

required. We use therefore an integral system. With this system one sorting gives all stars with  $N_s \geq x$  and  $N_h \leq y$  and two sortings suffice to give all stars with  $N_s = x$  and  $N_h = y$ .



information. There is a section left open in which can be inserted any information which may become necessary to include in the card.

About one hundred cards can be handled in a single selection process. Wearing of the cards does not seem to be prohibitive if the cards are of good quality. Cuts to be made near the corners have been avoided to prevent «hooking». The holes existing in three of the corners are not to be cut. They allow the extraction of the cards which

are wrongly orientated with respect to the stack.

Given the simplicity of the method, it is easy to change the model during the course of work. If the colour of the cards is also changed, the old ones can still be used. Although it is not essential to the usefulness of the system in a given laboratory, uniformity between laboratories would be an advantage. Information could be more easily exchanged, and cards could be made at a single centre, so reducing cost.

## Further Remarks on Tamm-Dancoff Method.

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(ricevuto il 2 Dicembre 1953)

In a recent note (with B. F. TOUSCHEK <sup>(1)</sup>) we have compared the exact scattering amplitude for a meson in the Wentzel-Blatt pair theory with the amplitude obtained with the first order T.D. method arriving to the conclusion that the T.D. method was in general a poor approximation to the exact solution.

The reason of limiting ourselves to the first order was of avoiding the coupled integral equations which are characteristic of the T.D. method in higher orders and which are generally known to lead to difficulties in the renormalization problem.

In order to exhibit these difficulties as well as some conclusions concerning the renormalization to which one can arrive by the aid of this simple example, we shall now study the second T.D. order. This is governed by the following equations in which both certain renormalization terms (to be simply eliminated by the  $\epsilon_k^{(1)}$  part of the counter-term) and the counter term itself have been left out and will be added at the end:

$$(1) \quad (\omega_k - E)u_k = -gf_k \sum_{k'} f_{k'} u_{k'} - \frac{1}{2} g \sum_{k_2, k_3} f_{k_2} f_{k_3} u_{kk_2k_3},$$

$$(2) \quad (\omega_{k_1} + \omega_{k_2} + \omega_{k_3} - E)u_{k_1k_2k_3} = -g[u_{k_1} f_{k_2} f_{k_3}]_{k_1k_2k_3} - g \sum_r f_r [f_{k_1} u_{rk_2k_3}]_{k_1k_2k_3}.$$

In these equations we have used, besides the symbols already used in <sup>(1)</sup> the following ones:  $f_k = v_k/\sqrt{2\omega_k}$ ;  $u_{k_1k_2k_3}$  = amplitude (symmetrical in  $k_1, k_2, k_3$ ) of the state with three mesons,  $[ ]_{k_1k_2k_3}$  = symmetrization with respect to  $k_1k_2k_3$ . One of the advantages of the present example with respect to this calculation is that the solution of (2) is exactly obtainable. The solution is (in the case  $E < 3\mu$ ):

$$u_{k_1k_2k_3} = -\frac{g}{2} \sum_{k_1, k_2, k_3} u_{k_1}^{-} f_{k_2}^{-} f_{k_3}^{-} \sum_{r_1 r_2 r_3} \frac{[W_{k_1}^{r_1} W_{k_2}^{r_2} W_{k_3}^{r_3}]_{r_1 r_2 r_3} \cdot W_{k_1}^{r_1} W_{k_2}^{r_2} W_{k_3}^{r_3}}{\omega_{r_1} + \omega_{r_2} + \omega_{r_3} - E}$$

<sup>(1)</sup> G. MORPURGO and B. F. TOUSCHEK: *Nuovo Cimento*, **10**, 1681 (1953). We would like to correct some oversights in that paper: formula (26) instead of the factor  $g(8/M^2)$  contains the factor  $(1/\pi)(M^2/8)$ ; the left side of the formula which follows (26) has to be multiplied by  $g$  and the center term divided by  $\pi$ ; the left side of the formula which precedes (27) and of (27) itself have to be multiplied by  $g\pi$  and  $g$  respectively.

where:

$$W_{k_i}^{r_i} = \delta_{k_i r_i} + g P \frac{f_{k_i} f_{r_i}}{\omega_{k_i} - \omega_{r_i}} \frac{1}{1 - g S_{r_i}^{(T.D.)}}$$

Putting  $\varphi_r = f_r (1 - g S_r^{(T.D.)})^{-1}$  we obtain the equation for  $u_k$ :

$$(3) \quad (\omega_k - E) u_k = - g f_k \sum_{k'} f_{k'} u_{k'} + \frac{g^2}{4} \sum_{k'} u_{k'} \sum_{r_1 r_2 r_3} \frac{[W_{k'}^{r_1} \varphi_{r_2} \varphi_{r_3}]_{r_1 r_2 r_3} \cdot W_{k'}^{r_1} \varphi_{r_2} \varphi_{r_3}}{\omega_{r_1} + \omega_{r_2} + \omega_{r_3}} \frac{1}{E}$$

If we expand  $u_k$  in terms of  $W_k^r$  according to:

$$u_k = \sum_r c_r W_k^r$$

equation (3) assumes a simpler form:

$$(4) \quad (\omega_r - E) c_r = \frac{g^2}{2} c_r \sum_{k, t} \frac{\varphi_k^2 \varphi_t^2}{\omega_r + \omega_k + \omega_t - E} + g^2 \sum_{r'} c_{r'} \varphi_{r'} \varphi_r \sum_t \frac{\varphi_t^2}{\omega_{r'} + \omega_r + \omega_t - E} + \\ + \frac{1}{2R} c_r \left( \sum_k \left\{ \frac{k}{\omega_k} \text{tg}^{-1} \frac{R g v_k^2}{2k(1 - g S_k^{(e)}) - \varepsilon_k^{(1)}} \right\} \right)$$

in which, to save writing, we have added the renormalization counter term which until now had been left out <sup>(2)</sup>; by the way we notice explicitly the other advantage of this example of knowing the exact counter term.

The first term on the right side of (4) has the character of a renormalization term; let's call this term  $c_r A$ . The important point is now that it has a completely different form from the last term (let's call it  $c_r B$ ) which we know to be the exact counter-term. If we don't want to expand  $A$  and  $B$  in series of  $g$  it is obviously impossible to renormalize eq. (4). If we expand  $A$  and  $B$  <sup>(3)</sup> we have to the second order in  $g$ :

$$c_r (A + B)^{(2)} = c_r \frac{g^2}{8} \left[ \sum_{k, t} \frac{v_k^2 v_t^2}{\omega_k \omega_t (\omega_r + \omega_k + \omega_t - E)} - \sum_{k, t} \frac{v_k^2 v_t^2}{\omega_k \omega_t (\omega_k + \omega_t)} \right] = \\ = - (\omega_r - E) c_r \frac{g^2}{8} \sum_{k, t} \frac{v_k^2 v_t^2}{\omega_k \omega_t} \frac{1}{(\omega_r + \omega_k + \omega_t - E)(\omega_k + \omega_t)}$$

so that, to order  $g^2$ , equation (4) has been renormalized.

But this is no more true at higher orders; for example at the third order the sum of:

$$c_r A^{(3)} = \frac{1}{4} g^3 c_r \sum_{k, t, s} \frac{v_k^2 v_t^2 v_s^2}{\omega_k \omega_t \omega_s (\omega_r + \omega_k + \omega_t - E)(\omega_k + \omega_s)}$$

<sup>(2)</sup> The  $\varepsilon_k^{(1)}$  part of it has to be subtracted, because, as we already said, we already made use of it in writing equation (1), (2).

<sup>(3)</sup> We must assume the possibility of expanding  $\text{tg}^{-1}$  in Taylor series around  $x=0$  which means  $x^2 < 1$ .

and

$$c_r B^{(3)} = \frac{1}{4} g^3 c_r \left[ \sum_{k,t,s} \frac{v_k^2 v_t^2 v_s^2}{\omega_k(\omega_k^2 - \omega_t^2)(\omega_k^2 - \omega_s^2)} - \sum_k \frac{R^2 v_k^6}{12k^2} \right]$$

does not give rise to an expression containing the factor  $c_r(E - \omega_r)$ .

We are thus faced with a situation in which we know the exact counter term from the beginning and we are not able however to renormalize the T.D. equations unless we use a series development and stop at the second order.

This example therefore teaches us that if we want to renormalize in the T.D. method we must use a Levy-Klein expansion in series of  $g$  and stop at a certain order; it is not possible to renormalize the T.D. method without making use of an expansion in series of  $g$ ; but this same necessity of using an expansion in series of  $g$  is one of the most weak points of the T.D. method, and in fact, together with the poor agreement of its first approximation with the exact solution, causes it to be a usually unreliable method for precise calculations (although it can give some qualitative insight).

## A proposito di un modello statistico a particelle indipendenti del nucleo pesante.

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(ricevuto l'8 Dicembre 1953)

In un modello statistico a particelle indipendenti del nucleo pesante più corretto di quello da noi usato in un recente lavoro <sup>(1)</sup>, per calcolare il cammino libero medio  $\lambda$  dei nucleoni nella materia nucleare, si tratterebbe di determinare innanzi tutto quale sia la configurazione del nucleo, e dedurne quindi per via quantisticamente corretta le probabilità di transizione che interessano.

Una approssimazione particolarmente semplice che si può fare in questa direzione è quella che consiste nell'assumere come configurazione del nucleo quella media calcolabile col metodo di WATANABE <sup>(2)</sup>, ma deducendo ancora, come già si è fatto in <sup>(1)</sup>, le probabilità di transizione direttamente dalle sezioni d'urto valevoli per nucleoni liberi. Si tratta evidentemente di una approssimazione che, quanto più basse sono le energie dei nucleoni di cui si considera l'urto, tanto meno può dare informazioni se non limitatamente ad ordini di grandezza: in tale senso queste ci sembrano degne di nota.

La configurazione media del nucleo pesante reale a temperatura  $T$  può essere

con buona approssimazione assimilata a quella di un gas di nucleoni senza interazione ad una temperatura fittizia più elevata  $T'$  <sup>(2,3)</sup>. Senza occuparci qui della esatta dipendenza di  $T'$  da  $T$ , ci interessa sapere che a  $T = 0$  si può fare

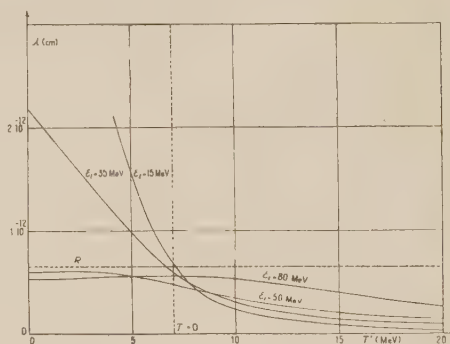


Fig. 1.

corrispondere un valore per  $T'$  dell'ordine di 7 MeV.

L'immagine che ne deriva è illustrata nelle figg. 1 e 2. La fig. 1 è ottenuta dalla fig. 3 di <sup>(1)</sup>, sostituendovi  $T'$  a  $T$  e introducendo per  $T$  una scala ancora indeterminata il cui zero sia a  $T' = 7$  MeV.

<sup>(1)</sup> A. KIND e G. PATERGNANI: *Nuovo Cimento*, **10**, 1375 (1953).

<sup>(2)</sup> S. WATANABE: *Zeits. f. Phys.*, **113**, 482 (1939).

<sup>(3)</sup> F. FERRARI e C. VILLI: *Nuovo Cimento*, **10**, 94 (1953).



In fig. 2 è riportata la trasparenza del nucleo di 100 nucleoni per un neutrone di energia incidente  $\varepsilon_0$ , calcolata con la configurazione usata in <sup>(1)</sup> (curva *a*) e con quella attuale (curva *b*).

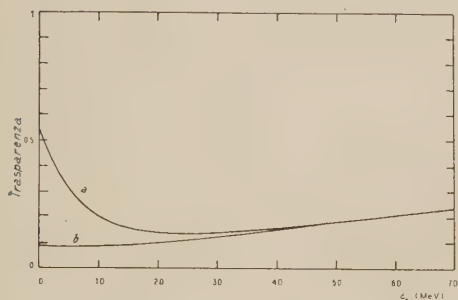


Fig. 2.

Le conclusioni di <sup>(1)</sup> per le reazioni ad alta energia ( $\varepsilon_0 > 50$  MeV) vengono ora essenzialmente riconfermate ed è immediato rendersi conto dalla fig. 1 in che verso si hanno le modificazioni.

In <sup>(1)</sup> non si sono considerate le reazioni nucleari prodotte da nucleoni incidenti la cui energia  $\varepsilon_0$  sia inferiore a 50 MeV. Come è noto, al di sotto di questo limite le reazioni possono essere descritte in modo generalmente soddisfacente, facendo uso della «ipotesi di

Bohr», ammettendo cioè che il nucleo sia praticamente opaco e che alla cattura si formi un cosiddetto «nucleo composto», che si porta in equilibrio prima di emettere i frammenti della reazione.

Dai valori della trasparenza dati in fig. 2 (curva *b*), da un lato, e da quelli di  $\lambda$  per  $T = 0$  di fig. 1 e del coefficiente di trasmissione  $P$  di fig. 1 in <sup>(1)</sup>, d'altro lato, si deduce che, secondo il modello ora considerato e dentro detti limiti di energia, il nucleo pesante deve risultare praticamente opaco e processi di emissione diretta non devono essere facilmente osservabili. La «ipotesi di Bohr» viene così ad essere dedotta direttamente dal modello a particelle indipendenti. È da notare che questa deduzione non sarebbe invece possibile con l'approssimazione usata in <sup>(1)</sup> (vedi fig. 2 curva *a*).

Osserviamo infine che il cammino libero medio di un nucleone avente energia cinetica di 15 MeV, valore medio in un nucleo a temperature non troppo alte, è, a tali temperature, dell'ordine di  $R$ : il che è una riprova, nell'approssimazione fatta, della coerenza del modello a particelle indipendenti, confermata dall'esperienza, nel descrivere il nucleo pesante in stati debolmente eccitati.

## Spiegazione classica del momento magnetico anomalo dell'elettrone.

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(ricevuto il 16 Dicembre 1953)

In una lettera precedente <sup>(1)</sup> abbiamo proposto un'equazione classica, relativisticamente invariante, per l'elettrone irraggiante basata sull'assunzione dell'esistenza di un intervallo di tempo elementare  $\tau_0$ .

Tale equazione può essere scritta:

$$(1) \quad \frac{m_0}{\tau_0} \left\{ u_\alpha(\tau) - u_\alpha(\tau - \tau_0) + \frac{u_\alpha(\tau)u_\beta(\tau)}{c^2} [u_\beta(\tau) - u_\beta(\tau - \tau_0)] \right\} = \frac{e}{c} F_{\alpha\beta} u_\beta,$$

o anche, tenendo conto che  $u_\alpha(\tau)u_\alpha(\tau) = -c^2$ :

$$(1') \quad -\frac{m_0}{\tau_0} \left[ u_\alpha(\tau - \tau_0) + \frac{u_\alpha(\tau)u_\beta(\tau)}{c^2} u_\beta(\tau - \tau_0) \right] = \frac{e}{c} F_{\alpha\beta} u_\beta.$$

Abbiamo poi mostrato che, assunta come origine dei tempi  $\tau = 0$ , le soluzioni dell'equazione precedenti in un istante  $\tau = n\tau_0$  risultano univocamente determinate assegnando il valore di due sole costanti arbitrarie e che dette soluzioni non sono mai del tipo « non fisico » come talvolta si verificava per l'equazione dell'elettrone di Dirac-Eliezer. Per definire univocamente le soluzioni negli istanti  $\tau = n\tau_0$  era necessario escludere le soluzioni periodiche dell'equazione:

$$(2) \quad u_\alpha(\tau) - u_\alpha(\tau - \tau_0) = 0$$

che possono sempre essere aggiunte a una soluzione particolare qualsiasi della (1).

Ci proponiamo ora di indagare se non è invece possibile attribuire un significato fisico a queste soluzioni periodiche che evidentemente si riferiscono, in quanto concernono il comportamento della particella a distanze inferiori a  $\sim R = (2/3)(e^2/m_0c^2)$ , a problemi di « struttura » dell'elettrone stesso.

Consideriamo allo scopo una particolare soluzione periodica della (2) e precisa-

<sup>(1)</sup> P. CALDIROLA: *Il Nuovo Cimento*, **10**, 1747 (1953).

mente la seguente:

$$(3) \quad \begin{cases} \dot{x} = -\beta c \sin 2\pi \frac{\tau}{\tau_0} \\ \dot{y} = \beta c \cos 2\pi \frac{\tau}{\tau_0} \\ \dot{z} = 0. \end{cases}$$

Integrando e ricordando che  $\tau_0 = 2R/c$ , si ha:

$$(4) \quad \begin{cases} x = \beta \frac{R}{\pi} \cos 2\pi \frac{c}{2R} \tau \\ y = \beta \frac{R}{\pi} \sin 2\pi \frac{c}{2R} \tau \\ z = 0. \end{cases}$$

Tali soluzioni corrispondono a un moto di rotazione dell'elettrone lungo un cerchio di raggio  $a = \beta R/\pi$  e con frequenza  $\nu = c/2R$ . Come è noto tale moto dà origine a un momento magnetico diretto lungo l'asse  $z$  e dato da:

$$(5) \quad \mu = \frac{1}{c} \pi a^2 e \nu = \frac{eR}{2\pi} \beta^2 = \frac{1}{3\pi} \frac{e^3}{m_0 c^2} \beta^2.$$

Ricordiamo ora che esperienze recenti <sup>(2)</sup> hanno provato l'esistenza di un momento magnetico anomalo  $\mu_a$  dell'elettrone a cui corrisponde, in ottimo accordo con le previsioni di SCHWINGER <sup>(3)</sup> un rapporto giromagnetico:

$$(6) \quad g_a \equiv -\frac{\mu_a}{s\mu_0} = 2 \frac{\alpha}{2\pi},$$

essendo  $s = 1/2$  lo spin (in unità  $\hbar$ ) dell'elettrone,  $\mu_0 = e\hbar/2m_0c$  il magnetone di Bohr e  $\alpha = e^2/\hbar c$  la costante di struttura fina. Il valore del momento magnetico anomalo risulta quindi:

$$(7) \quad \mu_a = \frac{1}{4\pi} \frac{e^3}{m_0 c^2}.$$

Confrontando la (5) con la (7) risulta che le due formule coincidono pur di prendere  $\beta^2 = 3/4$ .

Ricordando che  $m_0 = (2/3)(e^2/R)$  misura la massa effettiva dell'elettrone che compare nell'equazione della dinamica e che  $m_e = (1/2)(e^2/R)$  è generalmente assunta come valore della massa dell'elettrone associata all'energia elettrostatica, si ha la

<sup>(2)</sup> J. E. NAFE e E. B. NELSON: *Phys. Rev.*, **73**, 718 (1948).

<sup>(3)</sup> I. SCHWINGER: *Phys. Rev.*, **73**, 416 (1948).

notevole relazione:

$$(8) \quad \beta = \sqrt{\frac{m_e}{m_0}}.$$

Si osservi che l'energia cinetica  $E_{\text{cin}}$  associata al moto rotatorio (4) dell'elettrone, e cioè  $E_{\text{cin}} = m_0 c^2 \{1/\sqrt{1-\beta^2} - 1\}$  risulta esattamente eguale all'energia intrinseca  $m_0 c^2$  dell'elettrone: in altri termini la massa dell'elettrone nel suo moto « macroscopico », vale a dire nel moto determinato dalla (1) quando si prescinda dal moto « interno » descritto dalla soluzione periodica della (2), può essere associata alla energia cinetica dell'elettrone nel suo moto « interno ».

Questo fatto ci sembra di notevole interesse per la interpretazione fisica del modello di elettrone derivante dalla teoria da noi proposta.

Può sembrare a prima vista strano che la spiegazione del momento magnetico anomalo dell'elettrone, la quale viene giustamente considerata come uno dei più brillanti successi della odierna teoria quantistica dei campi, possa anche essere conseguita, come qui è stato mostrato, per mezzo di considerazioni classiche. La ragione di questo fatto appare però chiara se si osserva che nell'espressione (7) del momento magnetico anomalo, che deriva direttamente dalla formula (6) di Schwinger, non è contenuta la costante di Planck. Infatti da ciò si deve ritenere che l'esistenza del momento magnetico anomalo dell'elettrone non è un effetto essenzialmente quantistico. Tale è invece, come è ovvio, l'esistenza del momento magnetico normale  $\mu_0 = e\hbar/2m_0c$  per il quale sarebbe quindi illusorio tentare una spiegazione classica.



## LIBRI RICEVUTI E RECENSIONI

E. BLEULER and G. J. GOLDSMITH - *Experimental Nucleonics*. Pagine XVI + 393. Rinehart and Company, Inc., New York, 1952.

Come è detto dagli AA. nella prefazione, la scelta degli argomenti trattati nel libro deriva dalla triennale esperienza di un corso tenuto alla Purdue University.

Il metodo di esposizione della materia è analogo a quello per lo più seguito nei testi di esercitazioni di fisica adottati in ogni università, ma gli argomenti trattati sono specifici della fisica nucleare e tengono conto delle esigenze che si presentano spesso nei laboratori dove questo ramo della fisica è oggetto di ricerche.

In ciascun capitolo poche pagine, corredate da alcuni essenziali richiami bibliografici, hanno lo scopo di fornire allo studente le nozioni necessarie per intraprendere una determinata esperienza: successivamente lo sperimentatore viene guidato, si può dire passo passo, alla preparazione degli apparecchi e dei materiali occorrenti, all'esecuzione pratica dell'esperimento, all'esame dei risultati e alla loro interpretazione critica.

Ben poco è lasciato all'improvvisazione o all'iniziativa dello studente, al quale si vuole indicare un particolare procedimento corretto, magari da rielaborare e modificare opportunamente in casi che presentino analogie con quello trattato.

Alcuni capitoli introduttivi descrivono le apparecchiature di laboratorio di uso corrente nelle esperienze più avanti discusse (contatori di Geiger-Müller, circuiti registratori, sorgenti radioattive,

attrezzature di laboratorio chimico) e discutono il problema della protezione dalle radiazioni nucleari e delle relative dosi di tolleranza.

Il nucleo centrale del libro consta della trattazione, fatta nel modo prima accennato, di ventiquattro argomenti, suddivisi in tre gruppi.

Nella prima parte vengono discusse delle esperienze di carattere fondamentale in un laboratorio di fisica nucleare: uso di contatori di Geiger-Müller, taratura assoluta e relativa di sorgenti  $\beta$ , misure di assorbimento di  $\beta$  e  $\gamma$  e di produzione di isotopi radioattivi mediante bombardamento con neutroni lenti.

I metodi della radiochimica sono presentati nella seconda parte, e illustrati con opportuni esempi. Nozioni di chimica ed una certa pratica di laboratorio sono esplicitamente richieste per gli studenti, ma lo scopo delle esperienze qui suggerite è anche quello di informare i fisici su alcuni argomenti fondamentali della radiochimica, tanto da renderli capaci di seguire da vicino delle operazioni certamente assai interessanti per loro, come ad esempio la separazione di una sostanza attiva da un materiale precedentemente sottoposto ad irraggiamento.

Nella terza parte alcune delle tecniche proprie della fisica nucleare (misure di percorso di particelle  $\alpha$ , uso di spettrografi a cristallo, di lastre fotografiche e di metodi di coincidenza) sono applicate allo studio di decadimenti radioattivi e all'esame di spettri di energia e di correlazioni angolari tra particelle.

Data la varietà e la frammentarietà degli argomenti trattati, è certamente



impossibile dare un giudizio su ogni singolo capitolo.

Il libro non risente quasi del fatto di essere stato compilato tenendo conto dell'attrezzatura disponibile in un particolare istituto, in quanto si evita il più possibile di servirsi di apparecchiature o di strumenti complicati o poco usuali, e ciò lo rende senz'altro assai utile in ogni laboratorio discretamente attrezzato frequentato da studenti desiderosi di specializzarsi in fisica nucleare.

Le prime due parti del libro sono assai felici nella scelta degli argomenti e notevoli per semplicità e chiarezza; qualche riserva si può forse avanzare riguardo alla terza parte, in cui argomenti assai vasti sono discussi in poche pagine: in particolare appare lecito il dubbio che espe-

rienze della durata prevista di poche ore possano essere compiute soddisfacentemente in tale periodo di tempo, a meno che tutto il lavoro di messa a punto degli apparecchi non debba venire fatto in precedenza dai docenti; ciò non è però nelle intenzioni degli AA. e toglierebbe buona parte dell'utilità della esercitazione.

A parte considerazioni connesse col suo scopo immediato, il libro è certamente da considerarsi utilissimo anche come testo di consultazione da parte di un ricercatore in quanto può agevolargli molte ricerche bibliografiche, guidarlo nell'elaborazione di risultati sperimentali e fornirgli dei dati pratici non facilmente calcolabili o reperibili nella letteratura.

E. GERMAGNOLI

PROPRIETÀ LETTERARIA RISERVATA